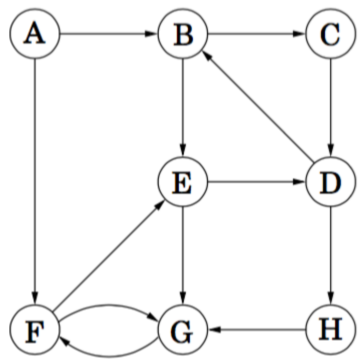
1 Graph Basics

In the first few parts, you will be answering questions on the following graph G.



(a) What are the vertex and edge sets V and E for graph G?

(b) Which vertex has the highest in-degree? Which vertex has the lowest in-degree? Which vertices have the same in-degree and out-degree?

(c) What are the paths from vertex B to F, assuming no vertex is visited twice? Which one is the shortest path?

(d) Which of the following are cycles in G?

i. $(B,C),(C,D),(D,B)$
ii. $(F,G),(G,F)$
iii. $(A,B),(B,C),(C,D),(D,B)$
iv. $(B,C),(C,D),(D,H),(H,G),(G,F),(F,E),(E,D),(D,B)$

(e) Which of the following are walks in G?

i. (E,G)
ii. $(E,G),(G,F)$
iii. $(F,G),(G,F)$
iv. $(A,B),(B,C),(C,D),(H,G)$
v. $(E,G),(G,F),(F,G),(G,C)$
vi. $(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)$

(f) Which of the following are tours in G?

i. (E,G)
ii. $(E,G),(G,F)$
iii. $(F,G),(G,F)$
iv. $(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)$

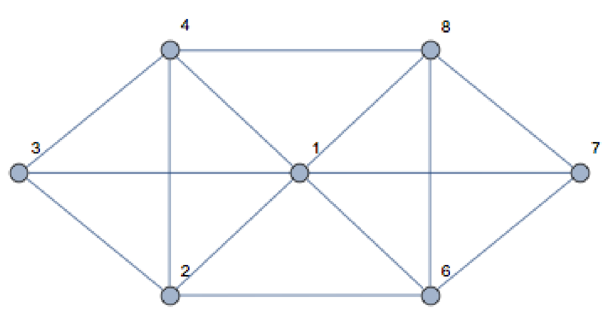
In the following three parts, let's consider a general undirected graph G with n vertices ($n \geq 3$).

(g) True/False: If each vertex of G has degree at most 1, then G does not have a cycle.

(h) True/False: If each vertex of G has degree at least 2, then G has a cycle.

(i) True/False: If each vertex of G has degree at most 2, then G is not connected.

2 Eulerian Tour and Eulerian Walk



(a) Is there an Eulerian tour in the graph above?

(b) Is there an Eulerian walk in the graph above?

(c) What is the condition that there is an Eulerian walk in an undirected graph? Briefly justify your answer.

3 Odd Degree Vertices

Claim: Let $G = (V, E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

Prove the claim above using:

(i) Direct proof (e.g., counting the number of edges in G). *Hint: in lecture, we proved that* $\sum_{v \in V} \deg v = 2|E|$.

(ii) Induction on $m = |E|$ (number of edges)

(iii) Induction on $n = |V|$ (number of vertices)

4 Trees

Recall that a *tree* is a connected acyclic graph (graph without cycles). In the note, we presented a few other definitions of a tree, and in this problem, we will prove two fundamental properties of a tree, and derive two definitions of a tree we learned from the note based on these properties. Let's start with the properties:

(a) Prove that any pair of vertices in a tree are connected by exactly one (simple) path.

(b) Prove that adding any edge (not already in the graph) between two vertices of a tree creates a simple cycle.

Now you will show that if a graph satisfies this property then it must be a tree:

(c) Prove that if the graph has no simple cycles and has the property that the addition of any single edge (not already in the graph) will create a simple cycle, then the graph is a tree.