1 Planarity

Consider graphs with the property T: For every three distinct vertices v_1, v_2, v_3 of graph G, there are at least two edges among them. Prove that if G is a graph on ≥ 7 vertices, and G has property T, then G is nonplanar.

2 Bipartite Graph

A bipartite graph consists of 2 disjoint sets of vertices (say L and R), such that no 2 vertices in the same set have an edge between them. For example, here is a bipartite graph (with $L = \{$green vertices$\}$ and $R = \{$red vertices$\}$), and a non-bipartite graph.

Figure 1: A bipartite graph (left) and a non-bipartite graph (right).

Prove that a graph has no tours of odd length if it is a bipartite (This is equivalent to proving that, a graph G being a bipartite implies that G has no tours of odd length).

3 Hypercubes

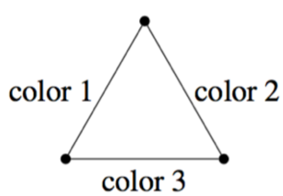
The vertex set of the n-dimensional hypercube $G = (V, E)$ is given by $V = \{0,1\}^n$ (recall that $\{0,1\}^n$ denotes the set of all n-bit strings). There is an edge between two vertices x and y if and only if x and y differ in exactly one bit position. These problems will help you understand hypercubes.

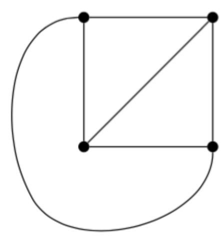
(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

(b) Show that for any $n \geq 1$, the n-dimensional hypercube is bipartite.

4 Edge Colorings

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.





(a) Show that the 4 vertex complete graph above can be 3 edge colored. (Use the numbers $1, 2, 3$ for colors. A figure is shown on the right.)

(b) Prove that any graph with maximum degree d can be edge colored with $2d - 1$ colors.

(c) Show that a tree can be edge colored with d colors where d is the maximum degree of any vertex.