1 Touring Hypercube

In the lecture, you have seen that if G is a hypercube of dimension n, then

- The vertices of G are the binary strings of length n.
- u and v are connected by an edge if they differ in exactly one bit location.

A *Hamiltonian tour* of a graph is a sequence of vertices $v_0, v_1, \dots, v_k$ such that:

- Each vertex appears exactly once in the sequence.
- Each pair of consecutive vertices is connected by an edge.
- v_0 and v_k are connected by an edge.

(a) Show that a hypercube has an Eulerian tour if and only if n is even. (*Hint: Euler's theorem*)

(b) Show that every hypercube has a Hamiltonian tour.

2 Divisible or Not

(a) Prove that for any number n, the number formed by the last two digits of n are divisible by 4 if and only if n is divisible by 4. (For example, '23xx' is divisible by 4 if and only if the number 'xx' is divisible by 4.)

(b) Prove that for any number n, the sum of the digits of n are divisible by 3 if and only if n is divisible by 3.

3 Modular Practice

(a) Calculate $72^{316} \bmod 7$.

(b) Solve the following system for x:

$$\begin{aligned} 3x &\equiv 4 + y \pmod{5} \\ 2(x-1) &\equiv 2y \pmod{5} \end{aligned}$$

(c) If it exists, find the multiplicative inverse of 31 mod 23 and 23 mod 31.

(d) What theorem allows us to know of the existence of multiplicative inverses?

(e) Prove the theorem in part (d).
(Hint: Remember an iff needs to be proven both directions. The gcd cannot be 0 or negative.)

4 Modular Arithmetic Equations

Solve the following equations for x and y modulo the indicated modulus, or show that no solution exists. Show your work.

(a) $9x \equiv 1 \pmod{11}$.

(b) $3x + 15 \equiv 4 \pmod{21}$.

(c) The system of simultaneous equations $3x + 2y \equiv 0 \pmod{7}$ and $2x + y \equiv 4 \pmod{7}$.