CS 70 Discrete Mathematics and Probability Theory
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DIS 3D

1 Countability Basics

1. Is $f : \mathbb{N} \to \mathbb{N}$, defined by $f(n) = n^2$ an injection (one-to-one)? Briefly justify.

2. Is $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^3 + 1$ a surjection (onto)? Briefly justify.

2 Count It!

For each of the following collections, determine and briefly explain whether it is finite, countably infinite (like the natural numbers), or uncountably infinite (like the reals):

(a) $\mathbb{N}$, the set of all natural numbers.

(b) $\mathbb{Z}$, the set of all integers.

(c) $\mathbb{Q}$, the set of all rational numbers.

(d) $\mathbb{R}$, the set of all real numbers.

(e) The integers which divide 8.

(f) The integers which 8 divides.

(g) The functions from $\mathbb{N}$ to $\mathbb{N}$.

(h) Computer programs that halt.

(i) Numbers that are the roots of nonzero polynomials with integer coefficients.

3 Hilbert's Paradox of the Grand Hotel

Consider a magical hotel with a countably infinite number of rooms numbered according to the natural numbers where all the rooms are currently occupied. Assume guests don't mind being moved out of their current room as long as they can get to their new room in a finite amount of time. In other words, guests can't be moved into a room that's infinitely far from the current one.

1. Suppose one new guest arrived in their car, how would you shuffle guests around to accommodate them? What if k guests arrived, where k is a constant positive integer?
2. Suppose a countably infinite number of guests arrived in an infinite length bus with seat numbers according to the natural numbers, how would you accommodate them?
3. Suppose a countably infinite number of infinite length buses arrive carrying countably infinite guests each, how would you accommodate them? (*Hint*: There are infinitely many prime numbers.)