

1 Condition on an Event

The random variable X has the PDF

$$f_X(x) = \begin{cases} cx^{-2}, & \text{if } 1 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Determine the value of c .
- (b) Let A be the event $\{X > 1.5\}$. Calculate $\mathbb{P}(A)$ and the conditional PDF of X given that A has occurred.

2 Uniform Distribution

You have two fidget spinners, each having a circumference of 10. You mark one point on each spinner as a needle and place each of them at the center of a circle with values in the range $[0, 10)$ marked on the circumference. If you spin both (independently) and let X be the position of the first spinner's mark and Y be the position of the second spinner's mark, what is the probability that $X \geq 5$, given that $Y \geq X$?

3 Chebyshev's Inequality vs. Central Limit Theorem

Let n be a positive integer. Let X_1, X_2, \dots, X_n be i.i.d. random variables with the following distribution:

$$\mathbb{P}[X_i = -1] = \frac{1}{12}; \quad \mathbb{P}[X_i = 1] = \frac{9}{12}; \quad \mathbb{P}[X_i = 2] = \frac{2}{12}.$$

(a) Calculate the expectations and variances of X_1 , $\sum_{i=1}^n X_i$, $\sum_{i=1}^n (X_i - \mathbb{E}[X_i])$, and

$$Z_n = \frac{\sum_{i=1}^n (X_i - \mathbb{E}[X_i])}{\sqrt{n/2}}.$$

(b) Use Chebyshev's Inequality to find an upper bound b for $\mathbb{P}[|Z_n| \geq 2]$.

(c) Can you use b to bound $\mathbb{P}[Z_n \geq 2]$ and $\mathbb{P}[Z_n \leq -2]$?

(d) As $n \rightarrow \infty$, what is the distribution of Z_n ?

(e) We know that if $Z \sim \mathcal{N}(0, 1)$, then $\mathbb{P}[|Z| \leq 2] = \Phi(2) - \Phi(-2) \approx 0.9545$. As $n \rightarrow \infty$, can you provide approximations for $\mathbb{P}[Z_n \geq 2]$ and $\mathbb{P}[Z_n \leq -2]$?

4 Arrows

You and your friend are competing in an archery competition. You are a more skilled archer than he is, and the distances of your arrows to the center of the bullseye are i.i.d. Uniform $[0, 1]$ whereas his are i.i.d. Uniform $[0, 2]$. To even out the playing field, you both agree that you will shoot one arrow and he will shoot two. The arrow closest to the center of the bullseye wins the competition. What is the probability that you will win? *Note: The distances from the center of the bullseye are uniform.*