Due: Sunday 7/7, 11:59 PM

Sundry

Before you start your homework, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Short Answer: Graphs

(a) (4 Points) Bob removed a degree 3 node in an n-vertex tree, how many connected components are in the resulting graph? (An expression that may contain n.)

(b) (4 Points) Given an n-vertex tree, Bob added 10 edges to it, then Alice removed 5 edges and the resulting graph has 3 connected components. How many edges must be removed to remove all cycles in the resulting graph? (An expression that may contain n.)

(c) (4 Points) True or False: For all $n \geq 3$, the complete graph on n vertices, K_n has more edges than the n-dimensional hypercube. Justify your answer.

(d) (4 Points) A complete graph with n vertices where n is an odd prime can have all its edges covered with x edge-disjoint Hamiltonian cycles (a Hamiltonian cycle is a cycle where each vertex appears exactly once). What is the number, x, of such cycles required to cover the a complete graph? (Answer should be an expression that depends on n.)

(e) (4 Points) Give a set of edge-disjoint Hamiltonian cycles that covers the edges of K_5, the complete graph on 5 vertices. (Each path should be a sequence (or list) of edges in K_5, where an edge is written as a pair of vertices from the set $\{0, 1, 2, 3, 4\}$ - e.g: $(0, 1), (1, 2)$.)

2 Bipartite Graph (15 points)

A bipartite graph consists of 2 disjoint sets of vertices (say L and R), such that no 2 vertices in the same set have an edge between them. For example, here is a bipartite graph (with $L =$

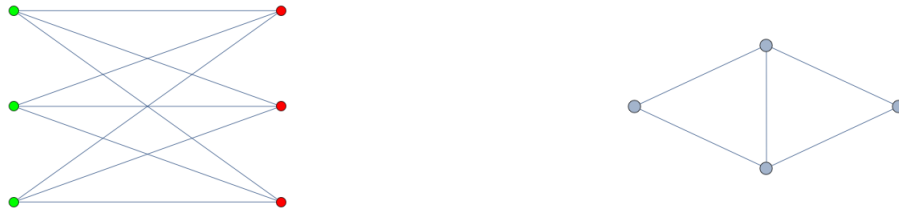{green vertices} and $R = \{$red vertices$\}$), and a non-bipartite graph.

Figure 1: A bipartite graph (left) and a non-bipartite graph (right).

In discussion 02A, we've proved that a graph has no tours of odd length if it is a bipartite. Now, please prove the reversed direction: If undirected G has no tours of odd length, it is a bipartite.

3 Triangulated Planar Graph

In this problem you will prove that every triangulated planar graph (every face has 3 sides; that is, every face has three edges bordering it, including the unbounded face) contains either (1) a vertex of degree 2, 3, 4 (Note that a triangulated planar graph cannot contain a vertex with degree 1; can you think about a reason why?); (2) two degree 5 vertices which are adjacent; or (3) a degree 5 and a degree 6 vertices which are adjacent. We will construct a proof by cases. Note that while we are attempting to prove that the former holds, we are still assuming that the graph can have vertices with degree greater than 6. Please justify your answers.

(a) (5 points) Place a "charge" on each vertex v of value $6 - \text{degree}(v)$. What is the sum of the charges on all the vertices? (*Hint*: Use Euler's formula and the fact that the planar graph is triangulated.)

(b) (5 points) What is the charge of a degree 5 vertex and of a degree 6 vertex?

(c) (5 points) Suppose now that we shift $1/5$ of the charge of a degree 5 vertex to each of its neighbors that has a negative charge. (We refer to this as "discharging" the degree 5 vertex.) After discharging all degree 5 vertices, when would there be a degree 5 vertex with positive remaining charge?

(d) (5 points) If no degree 5 vertices have positive charge after discharging the degree 5 vertices, does there exist any vertex with positive charge after discharging? If there is such a vertex, what are the possible degrees of that vertex?

(e) (5 points) Suppose there exists a degree 7 vertex with positive charge after discharging the degree 5 vertices. How many neighbors of degree 5 might it have?

(f) (5 points) Continuing from Part (e). Since the graph is triangulated, are two of these degree 5 vertices adjacent?

(g) (5 points) Finish the proof from the facts you obtained from the previous parts.

4 Modular Exponentiation

Compute the following:

(a) (5 points) $13^{2018} \pmod{12}$

(b) (5 points) $8^{11111} \pmod 9$

(c) (5 points) $7^{256} \pmod{11}$

(d) (5 points) $3^{160} \pmod{23}$

5 Euler's Totient Function

Euler's totient function is defined as follows:

$$\phi(n) = |\{i : 1 \le i \le n, \gcd(n, i) = 1\}|$$

In other words, $\phi(n)$ is the total number of positive integers less than or equal to n which are relatively prime to it. Here is a property of Euler's totient function that you can use without proof:

For m, n such that $\gcd(m, n) = 1$, $\phi(mn) = \phi(m) \cdot \phi(n)$.

(a) (5 points) Let p be a prime number. What is $\phi(p)$?

(b) (5 points) Let p be a prime number and k be some positive integer. What is $\phi(p^k)$?

(c) (5 points) Let p be a prime number and a be a positive integer smaller than p. What is $a^{\phi(p)} \pmod p$?
    (Hint: use Fermat's Little Theorem.)

(d) (5 points) Let b be a positive integer whose prime factors are $p_1, p_2, \ldots, p_k$. We can write $b = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$.

Show that for any a relatively prime to b, the following holds:

$$\forall i \in \{1, 2, \ldots, k\}, \quad a^{\phi(b)} \equiv 1 \pmod{p_i}$$

6 Just a Little Proof (15 points)

Suppose that p and q are distinct odd primes and a is an integer such that $\gcd(a, pq) = 1$. Prove that $a^{(p-1)(q-1)+1} \equiv a \pmod{pq}$.