Due: Sunday July 21, 2019 at 11:59 PM

Sundry

Before you start your homework, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Kolmogorov Complexity

Compression of a bit string x of length n involves creating a program shorter than n bits that returns x. The Kolmogorov complexity of a string $K(x)$ is the length of shortest program that returns x (i.e. the length of a maximally compressed version of x).

(a) Explain why "the smallest positive integer not definable in under 100 characters" is paradoxical.

(b) Prove that for any length n, there must be at least one bit string that cannot be compressed to fewer than n bits.

(c) Imagine you had the program K, which outputs the Kolmogorov complexity of string. Design a program P that when given integer n outputs the bit string of length n with the highest Kolmogorov complexity. If there are multiple strings with the highest complexity, output the lexicographically first (i.e. the one that would come first in a dictionary).

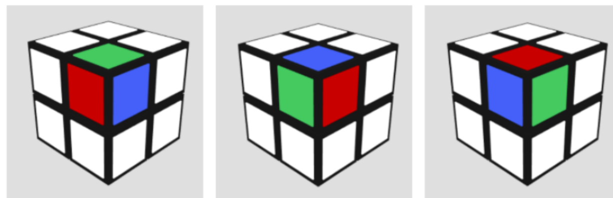
(d) Suppose the program P you just wrote can be written in m bits. Show that P and by extension, K, cannot exist, for a sufficiently large input n.

2 Rubik's Cube Scrambles

We wish to count the number of ways to scramble a $2 \times 2 \times 2$ Rubik's Cube, and take a quick look at the $3 \times 3 \times 3$ cube. Leave your answer as an expression (rather than trying to evaluate it to get a specific number).

(a) The $2 \times 2 \times 2$ Rubik's Cube is assembled from 8 "corner pieces" arranged in a $2 \times 2 \times 2$ cube. How many ways can we assign all the corner pieces a position?

(b) Each corner piece has three distinct colors on it, and so can also be oriented three different ways once it is assigned a position (see figure below). How many ways can we *assemble* (assign each piece a position and orientation) a $2 \times 2 \times 2$ Rubik's Cube?



Three orientations of a corner piece

(c) The previous part assumed we can take apart pieces and assemble them as we wish. But certain configurations are unreachable if we restrict ourselves to just turning the sides of the cube. What this means for us is that if the orientations of 7 out of 8 of the corner pieces are determined, there is only 1 valid orientation for the eighth piece. Given this, how many ways are there to *scramble* (as opposed to *assemble*) a $2 \times 2 \times 2$ Rubik's Cube?

(d) We decide to treat scrambles that differ only in overall positioning (in other words, the entire cube is flipped upside-down or rotated but otherwise unaltered) as the same scramble. Then we overcounted in the previous part! How does this new condition change your answer to the previous part?

(e) Now consider the $3 \times 3 \times 3$ Rubik's Cube. In addition to 8 corner pieces, we now have 12 "edge" pieces, each of which can take 2 orientations. How many ways can we *assemble* a $3 \times 3 \times 3$ Rubik's Cube?

3 The Count

(a) How many permutations of COSTUME contain "COME" as a substring? How about as a subsequence (meaning the letters of "COME" have to appear in that order, but not necessarily next to each other)?

(b) How many of the first 100 positive integers are divisible by 2, 3, or 5?

(c) How many ways are there to choose five nonnegative integers x_0, x_1, x_2, x_3, x_4 such that $x_0 + x_1 + x_2 + x_3 + x_4 = 100$, and $x_i \equiv i \pmod 5$?

(d) The Count is trying to choose his new 7-digit phone number. Since he is picky about his numbers, he wants it to have the property that the digits are non-increasing when read from left to right. For example, 9973220 is a valid phone number, but 9876545 is not. How many choices for a new phone number does he have?

4 Binomial Beads

(a) Satish is making school spirit keychains, which consist of a sequence of n beads on a string. He has blue beads and gold beads. How many unique keychains can he make with exactly $k \leq n$ blue beads?

(b) Satish decides to sell his keychains! He decides on the following pricing scheme:

- Blue beads have a value of x
- Gold beads have a value of y
- The price of a keychain is the product of the values of all of its beads.

What is the price of a keychain with exactly $k \leq n$ blue beads?

(c) Satish decides to make exactly one of every possible unique keychain. If he sells every keychain he creates, how much revenue will he make? Use parts (a) and (b), and leave your answer in summation form.

(d) Draw a connection between part (c) and the binomial theorem.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Hint: How do you calculate the product (x + y)(x + y)?

5 Probability Practice

(a) If we put 5 math, 6 biology, 8 engineering, and 3 physics books on a bookshelf at random, what is the probability that all the math books are together?

(b) A message source M of a digital communication system outputs a word of length 8 characters, with the characters drawn from the ternary alphabet $\{0,1,2\}$, and all such words are equally probable. What is the probability that M produces a word that looks like a byte (*i.e.*, no appearance of '2')?

(c) If five numbers are selected at random from the set $\{1,2,3,\ldots,20\}$, what is the probability that their minimum is larger than 5? (A number can be chosen more than once, and the order in which you select the numbers matters)

6 Shooting Range

You and your friend are at a shooting range. You ran out of bullets. Your friend still has two bullets left but magically lost his gun. Somehow you both agree to put the two bullets into your six-chambered revolver in successive order, spin the revolver, and then take turns shooting. Your first shot was a blank. You want your friend to shoot a blank too; should you spin the revolver again before you hand it to your friend?

7 Cliques in Random Graphs

Consider a graph $G = (V, E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v, we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads. So for example if $n = 2$, then with probability $1/2$, $G = (V, E)$ is the graph consisting of two vertices connected by an edge, and with probability $1/2$ it is the graph consisting of two isolated vertices.

(a) What is the size of the sample space?

(b) A k-clique in graph is a set of k vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example a 3-clique is a triangle. What is the probability that a particular set of k vertices forms a k-clique?

(c) Prove that the probability that the graph contains a k-clique, for $k \geq 4 \log n + 1$, is at most $1/n$. (The log is taken base 2).

Hint: Apply the union bound and part (c).