Due: Sunday July 28, 2019 at 11:59 PM

Sundry

Before you start your homework, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Combinatorial Proof!

Prove that for $0 < k < n$, $\binom{n}{k} = \sum_{i=0}^{k} \binom{n-i-1}{k-i}$.

2 Cliques in Random Graphs

In last week's homework you worked on a graph $G = (V,E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v, we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads. Now consider:

(a) Prove that $\binom{n}{k} \le n^k$. *Optional:* Can you come up with a combinatorial proof? Of course, an algebraic proof would also get full credit.

(b) Prove that the probability that the graph contains a k-clique, for $k \ge 4\log n + 1$, is at most $1/n$. (The log is taken base 2). *Hint:* Apply the union bound and part (a).

3 Balls and Bins, All Day Every Day

You throw n balls into n bins uniformly at random, where n is a positive *even* integer.

(a) What is the probability that exactly k balls land in the first bin, where k is an integer $0 \le k \le n$?

(b) What is the probability p that at least half of the balls land in the first bin? (You may leave your answer as a summation.)

(c) Using the union bound, give a simple upper bound, in terms of p, on the probability that some bin contains at least half of the balls.

(d) What is the probability, in terms of p, that at least half of the balls land in the first bin, or at least half of the balls land in the second bin?

(e) After you throw the balls into the bins, you walk over to the bin which contains the first ball you threw, and you randomly pick a ball from this bin. What is the probability that you pick up the first ball you threw? (Again, leave your answer as a summation.)

4 Indicator Variables

(a) After throwing n balls into m bins at random, what is the expected number of bins that contains exactly k balls?

(b) Alice and Bob each draw k cards out of a deck of 52 distinct cards with replacement. Find k such that the expected number of common cards that both Alice and Bob draw is at least 1.

(c) How many people do you need in a room so that you expect that there is going to be a shared birthday on a Monday of the year (assume 52 Mondays in a year and 365 days in a year)? By "expect" we mean on expectation, there should be 1 pair of people that have a shared birthday.

5 Poisoned Smarties

Supposed there are 3 men who are all owners of their own Smarties factories. Burr Kelly, being the brightest and most innovative of the men, produces considerably more Smarties than his competitors and has a commanding 45% of the market share. Yousef See, who inherited his riches, lags behind Burr and produces 35% of the world's Smarties. Finally Stan Furd, brings up the rear with a measly 20%. However, a recent string of Smarties related food poisoning has forced the FDA investigate these factories to find the root of the problem. Through his investigations, the inspector found that one Smarty out of every 100 at Kelly's factory was poisonous. At See's factory, 1.5% of Smarties produced were poisonous. And at Furd's factory, the probability a Smarty was poisonous was 0.02.

(a) What is the probability that a randomly selected Smarty will be safe to eat?

(b) If we know that a certain Smarty didn't come from Burr Kelly's factory, what is the probability that this Smarty is poisonous?

(c) Given this information, if a randomly selected Smarty is poisonous, what is the probability it came from Stan Furd's Smarties Factory?

6 Testing Model Planes

Dennis is testing model airplanes. He starts with n model planes which each independently have probability p of flying successfully each time they are flown, where $0 < p < 1$. Each day, he flies every single plane and keeps the ones that fly successfully (i.e. don't crash), throwing away all other models. He repeats this process for many days, where each "day" consists of Dennis flying any remaining model planes and throwing away any that crash. Let X_i be the random variable representing how many model planes remain after i days. Note that $X_0 = n$. Justify your answers for each part.

(a) What is the distribution of X_1? That is, what is $\mathbb{P}[X_1 = k]$?

(b) What is the distribution of X_2? That is, what is $\mathbb{P}[X_2 = k]$? Show that X_2 follows a binomial distribution by finding some n' and p' such that $X_2 \sim \text{Binom}(n', p')$.

(c) Repeat the previous part for X_t for arbitrary $t \geq 1$.

(d) What is the probability that at least one model plane still remains (has not crashed yet) after t days? Do not have any summations in your answer.

(e) Considering only the first day of flights, is the event A_1 that the first and second model planes crash independent from the event B_1 that the second and third model planes crash? Recall that two events A and B are independent if $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$. Prove your answer using this definition.

(f) Considering only the first day of flights, let A_2 be the event that the first model plane crashes *and* exactly two model planes crash in total. Let B_2 be the event that the second plane crashes on the first day. What must n be equal to in terms of p such that A_2 is independent from B_2? Prove your answer using the definition of independence stated in the previous part.

(g) Are the random variables X_i and X_j, where $i < j$, independent? Recall that two random variables X and Y are independent if $\mathbb{P}[X = k_1 \cap Y = k_2] = \mathbb{P}[X = k_1]\mathbb{P}[Y = k_2]$ for all k_1 and k_2. Prove your answer using this definition.

7 Geometric Distribution

Two faulty machines, M_1 and M_2, are repeatedly run synchronously in parallel (i.e., both machines execute one run, then both execute a second run, and so on). On each run, M_1 fails with probability p_1 and M_2 fails with probability p_2, all failure events being independent. Let the random variables X_1, X_2 denote the number of runs until the first failure of M_1, M_2 respectively; thus X_1, X_2 have geometric distributions with parameters p_1, p_2 respectively. Let X denote the number of runs until the first failure of *either* machine.

(a) Show that X also has a geometric distribution, with parameter $p_1 + p_2 - p_1 p_2$.

(b) Now, two technicians are hired to check on the machines every run. They decide to take turns checking on the machines every run. What is the probability that the first technician is the first one to find a faulty machine?