1 Review of Sets

A **set** is a well defined collection of objects. These objects are called **elements** or **members** of the set, and they can be anything, including numbers, letters, people, cities, and even other sets. By convention, sets are usually denoted by capital letters and can be described or defined by listing its elements and surrounding the list by curly braces. For example, we can describe the set A to be the set whose members are the first five prime numbers, or we can explicitly write: $A = \{2, 3, 5, 7, 11\}$.

If x is an element of A, then we write $x \in A$. Similarly, if y is not an element of A, then we write $y \notin A$. Thus, given A as defined in the previous paragraph, we would have $5 \in A$, but $10 \notin A$. Two sets A and B are said to be **equal**, written as $A = B$, if they have the same elements. The order and repetition of elements do not matter, so {red, white, blue} = {blue, white, red} = {red, white, white, blue}.

Sometimes, we will use "set builder notation" to define more complicated sets. As an example, the set of all rational numbers, denoted by $\mathbb{Q}$, can be written as $\{\frac{a}{b} \mid a, b \text{ are integers}, b \neq 0\}$. In English, this is read as "the set of all fractions such that the numerator is an integer and the denominator is a non-zero integer". The general form of this notation is { item | condition on that item }, and is interpreted to mean "the set of all items which satisfy this condition".

1.1 Cardinality

We can also talk about the size of a set, or its **cardinality**. If $A = \{1,2,3,4\}$, then the cardinality of A, denoted by $|A|$, is 4. If A is a finite set[1], it's cardinality must be a non-negative integer; furthermore, there is a unique set with cardinality zero. This set is called the **empty set** and is denoted as $\emptyset$.

1.2 Subsets and Proper Subsets

If every element of a set A is also in set B, then we say that A is a **subset** of B, written $A \subseteq B$. Equivalently we can write $B \supseteq A$, or B is a superset of A. A **proper subset** of B is a set A such that $A \subseteq B$ but $A \neq B$, meaning that A excludes at least one element of B. This is denoted $A \subset B$. For example, consider the set $B = \{1,2,3,4,5\}$. Then $\{1,2,3\}$ is both a subset and a proper subset of B, while $\{1,2,3,4,5\}$ is a subset but not a proper subset of B. Here are a few basic properties regarding subsets:

- The empty set is a proper subset of any nonempty set A: $\emptyset \subset A$.
- The empty set is a subset of every set B: $\emptyset \subseteq B$.
- Every set A is a subset of itself: $A \subseteq A$.

[1] We'll discuss the cardinalities of infinite sets later in the class. For now, we'll just say their cardinality is "infinite".

1.3 Intersections and Unions

The **intersection** of a set A with a set B, written as $A \cap B$, is the set containing all elements which are in both A and B. Two sets are said to be **disjoint** if $A \cap B = \emptyset$. The **union** of a set A with a set B, written as $A \cup B$, is the set of all elements which are in either A or B or both. For example, if A is the set of all positive even numbers, and B is the set of all positive odd numbers, then $A \cap B = \emptyset$, and $A \cup B = \mathbb{Z}^+$, or the set of all positive integers. Here are a few properties of intersections and unions:

- $A \cup B = B \cup A$
- $A \cup \emptyset = A$
- $A \cap B = B \cap A$
- $A \cap \emptyset = \emptyset$

1.4 Relative Complements

If A and B are two sets, then the **relative complement** of A in B, also known as the **set difference** between B and A, is the set of elements in B, but not in A: $\{x \in B \mid x \notin A\}$. This is generally denoted as $B - A$ or sometimes as $B \backslash A$. For example, if $B = \{1,2,3\}$ and $A = \{3,4,5\}$, then $B - A = \{1,2\}$. For another example, if $\mathbb{R}$ is the set of real numbers and $\mathbb{Q}$ is the set of rational numbers, then $\mathbb{R} - \mathbb{Q}$ is the set of irrational numbers. Here are some important properties of complements:

- $A - A = \emptyset$
- $A - \emptyset = A$
- $\emptyset - A = \emptyset$

Note that unlike with intersections and unions, it is generally not the case that $A - B = B - A$.

1.5 Significant Sets

In mathematics, some sets are referred to so commonly that they are denoted by special symbols. These include:

- $\mathbb{N}$ denotes the set of all natural numbers: $\{0,1,2,3,...\}$. Note here that while some classes define $\mathbb{N}$ to not include 0, we will always assume it does for the purposes of this class.
- $\mathbb{Z}$ denotes the set of all integer numbers: $\{\dots,-2,-1,0,1,2,\dots\}$.
- $\mathbb{Q}$ denotes the set of all rational numbers: $\{\frac{a}{b} \mid a,b \in \mathbb{Z},\, b \neq 0\}$.
- $\mathbb{R}$ denotes the set of all real numbers.
- $\mathbb{C}$ denotes the set of all complex numbers.

1.6 Products and Power Sets

The **Cartesian product** (also called the **cross product**) of two sets A and B, written as $A \times B$, is the set of all pairs whose first component is an element of A and whose second component is an element of B. In set notation, $A \times B = \{(a,b) \mid a \in A, b \in B\}$. For example, if $A = \{1,2,3\}$ and $B = \{u,v\}$, then $A \times B = \{(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)\}$. And $\mathbb{N} \times \mathbb{N} = \{(0,0),(1,0),(0,1),(1,1),(2,0),\ldots\}$ is the set of all pairs of natural numbers.

Given a set S, the **power set** of S, denoted by $\mathscr{P}(S)$, is the set of all subsets of S: $\{T \mid T \subseteq S\}$. For example, if $S = \{1,2,3\}$, then the power set of S is: $\mathscr{P}(S) = \{\{\},\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$. Note that, if $|S| = k$, then $|\mathscr{P}(S)| = 2^k$. [Do you see why?]

2 Review of Mathematical Notation

2.1 Sums and Products

There is a compact notation for writing sums or products of large numbers of items. For example, to write $1+2+\cdots+n$, without having to say "dot dot dot", we can write $\sum_{i=1}^{n} i$. More generally we can write the sum $f(m)+f(m+1)+\cdots+f(n)$ as $\sum_{i=m}^{n} f(i)$. Thus, for example, $\sum_{i=5}^{20} i^2 = 5^2+6^2+\cdots+20^2$.

Analogously, to write the product $f(m)f(m+1)\cdots f(n)$ we use the notation $\prod_{i=m}^{n} f(i)$. As an example, $\prod_{i=1}^{n} i = 1 \cdot 2 \cdots n$ is the product of the first n positive integers.

2.2 Universal and Existential Quantifiers

When writing propositions, we often want to say that something is true for *all* elements of some set or alternatively for *some* unspecified element of the set. In the former case, we use the quantifier $\forall$ (read "for all"), while in the latter case we would use $\exists$ (read "there exists"). As an example, if we wanted to make the (false) claim that all natural numbers were primes, we would write $(\forall n \in \mathbb{N})(n \text{ is prime})$. If we wished instead to make the more reasonable assertion that there exists a prime number, we would write $(\exists n \in \mathbb{N})(n \text{ is prime})$.

As a special case, note that for any claim C, the statement $(\exists x \in \emptyset)(C(x))$ is considered false, as there is no x in the empty set at all, much less one which satisfies C. Conversely, $(\forall x \in \emptyset)(C(x))$ is considered true, as there is no x in the empty set which could possibly fail to satisfy C. In this second case, we often say that our statement is **vacuously** or **trivially** true.