Bonus Note 3

This note is adapted from Chapter 6 of "Elements of Set Theory" by Herbert Enderton.

1 Cantor-Schroder-Bernstein Theorem

In note 10, we stated and used the following theorem without proof:

Theorem B3.1. *Let A and B be sets. If there exist one-to-one functions $f : A \to B$ and $g : B \to A$, there is a bijection $b : A \to B$.*

In this note, we give a formal proof of this fact.

1.1 Intuition

Before diving into the proof itself, we should get some intuition about how we are going to build the bijection b. The statement of Theorem B3.1 gives tells us nothing about A and B other than that we have the two functions f and g. Thus, at least intuitively, if we are to have any hope of constructing b, we will need to somehow base it off f and g.

One way we could attempt to do this would be to start with f and somehow fix the fact that it is not onto. While this may be possible, it seems intuitively difficult to do so — after all, if we try to map an element $x \in A$ to some element of B not hit by f, we now are no longer hitting $f(x)$. Instead, as we will see, it is much easier to start from a function which is onto but not one-to-one.

To this end, we start by considering g. Let R_g be the set of all elements mapped to by g; that is, $R_g = \{g(x) | x \in B\}$. If we interpret g as a function from B to R_g (instead of to A), it will certainly then be onto — and changing the range won't affect the fact that it is onto. Thus, g as a function from B to R_g is a bijection, meaning we have an inverse $g^{-1} : R_g \to B$.

This function g^{-1} is almost, but not quite, what we want. We were looking for a bijection from A to B, and instead got a bijection from some subset of A to B. In order to extend this into a function with the proper domain, we need to decide where to map the elements in $A - R_g$, which we will denote A_0 for brevity. Where could we possibly map these elements? g is not helpful, as it doesn't touch them at all. Thus, the only option left to us is to use f. This leads us to our first attempt at a bijection:

$$b(x) = \begin{cases} f(x) & x \in A_0 \\ g^{-1}(x) & x \in R_g \end{cases}$$

This is a good start in that it is in fact a function from A to B, and with some close inspection, one can verify that it will be onto. However, it will not be one-to-one. There cannot be any collisions between evaluations of f or between two evaluations of g^{-1} (as they are both individually one-to-one), but we will run into issues if there is an $x \in A_0$ and an $x' \in R_g$ such that $f(x) = g^{-1}(x')$. In other words, we have an issue if $g(f(x)) = x'$ for some $x \in A_0$ and $x' \in R_g$. We have no other obvious place to map x, so the only option left to us is to displace x'.

This will now lead us to our second attempt at creating a bijection. We first define $A_1 = \{g(f(x)) | x \in A_0\}$, which is precisely the set of elements in R_g that need to be displaced. Since we can no longer apply g^{-1} to the elements in A_1, again the only reasonable option we have remaining is to apply f instead. This gives us

$$b(x) = \begin{cases} f(x) & x \in (A_0 \cup A_1) \\ g^{-1}(x) & \text{otherwise} \end{cases}$$

This is progress in the right direction, as we've dealt with the collisions between elements of A_0 and A_1. However, we've now created collisions between elements of A_1 (to which we're applying f) and those in $A_2 := \{g(f(x)) | x \in A_1\}$ (to which we're applying g^{-1}). We thus are forced to displace everything in A_2 and apply f instead of g^{-1} to them. This then in turn creates its own collisions with $A_3 := \{g(f(x)) | x \in A_2\}$, and so the cycle continues.

It might seem at this point like we are stuck: every time we fix some collisions, we create others, so there's no way we can ever have no collisions. This would indeed be true if we stopped after some finite number of steps. However, the key to the proof of the Cantor-Scrhöder-Bernstein Theorem is that we can do this fixing step *infinitely many times*. Each fixing step repairs the collisions of the one before, so as long as there's no "last" step, we'll never have any collisions. We can now formalize this intuition into a full proof.

1.2 Proof

Proof of Theorem B3.1. Suppose we are given one-to-one functions $f : A \to B$ and $g : B \to A$. Letting $R_g = \{g(x) | x \in B\}$, we can as in the previous section find a bijection $g^{-1} : R_g \to B$. Letting $A_0 = A - R_g$ and $A_i = \{g(f(x)) | x \in A_{i-1}\}$ for $i \geq 1$, we define

$$b(x) = \begin{cases} f(x) & x \in A_n \text{ for some } n \\ g^{-1}(x) & \text{otherwise} \end{cases}$$

In order to complete the proof, we simply need to show that b is a bijection, which we will do by showing that it is one-to-one and onto.

Onto: We need to show that for any $y \in B$, there is an $x \in A$ such that $b(x) = y$. We have two cases to consider. The first is if $g(y)$ is in some A_n. We know that $g(y) \in R_g$ and hence is not in A_0, so we must have that $n \geq 1$. Hence, by the definition of A_n for $n \geq 1$, we must have an $x' \in A_{n-1}$ such that $g(f(x')) = g(y)$. But g is one-to-one, so this is only possible if $f(x') = y$. Since $x' \in A_{n-1}$, we have that $h(x') = f(x') = y$, so we have indeed found an element of A that maps to y.

The only case left to consider is if $g(y)$ is not in A_n for any n. In this case, by our definition of b, we know that $b(g(y)) = g^{-1}(g(y)) = y$, so we've again found an element of A that maps to y. Hence, g is onto.

One-to-one: We next need to show that it is impossible to have $b(x) = b(x')$ for $x \neq x'$. Suppose for contradiction that there are some $x \neq x'$ with $b(x) = b(x')$. It cannot be the case that $b(x) = f(x)$ and $b(x') = f(x')$ as f is an injective function, so we can't have $f(x) = f(x')$. Similarly, g^{-1} is a bijection from R_g to B (and hence in particular is one-to-one), so it is impossible for $b(x) = g^{-1}(x)$ and $b(x') = g^{-1}(x')$. The only possibility left, then, is that one of the two inputs falls under the first case of our definition of b, while the other point falls under the second case.

Suppose without loss of generality that x falls under the first case and x' falls under the second. In other words, we have that $x \in A_n$ for some n, but x' is not in any A_i. In order for $b(x)$ to equal $b(x')$, we must have that $f(x) = g^{-1}(x')$, meaning that $g(f(x)) = x'$. But then since $x \in A_n$, we must have that $x' \in A_{n+1}$, contradicting the fact that x' is not in any of the A_is! Thus, we can conclude that b must be one-to-one. □

1.3 An Example

As a final part of this note, let's see the kind of bijection we get from the Cantor-Schröder-Bernstein Theorem. In particular, we will use as our case study $A = B = \mathbb{N}$ with the injections $f(x) = g(x) = 2x$.

We first must define our set A_0. This is precisely the set of values in $\mathbb{N}$ which are not of the form $2k$, ie, the odd numbers. In order to get our set A_1, we apply $g(f(x)) = 4x$ to every element of A_0. This gives us the set of all numbers which are divisible by 4 but have no factors of two beyond that; that is, we get $A_1 = \{4o | o \text{ is odd}\}$. Applying $g(f(x))$ to all elements in this set, we get $A_2 = \{16o | o \text{ is odd}\}$.

If we keep applying this pattern, we will notice that $A_i = \{2^{2i}o | o \text{ is odd}\}$.

Exercise. Prove this formula for A_i by induction.

What this tells us is that a natural number x is in some A_i if its prime factorization contains an even number of factors of 2. In other words, x is in some A_i if there exists a natural number n and an odd number o such that $x = 2^{2n}o$. Thus, the Cantor-Schröder Bernstein Theorem gives us the following bijection from $\mathbb{N}$ to $\mathbb{N}$:

$$b(x) = \begin{cases} 2x & x = 2^{2n}o \text{ for some } n \in \mathbb{N} \text{ and odd } o \\ \frac{x}{2} & \text{otherwise} \end{cases}$$

Sanity check! Verify that the above function is a bijection.
