Chapter 1

Beginnings

In this chapter we'll go over a few fundamental definitions and concepts!

1.1 (Sorta) How Math Works

Since this may be your first math class with a focus on proofs and rigor, I thought it might be helpful to do a high-level overview of the field of mathematics.

In any mathematical system, one starts with the following:

1. **Primitive notions**, or undefined concepts. This is stuff that you and everybody has a general agreed-upon idea of but everybody also agreed that there was no point [1] trying to precisely define because the concept is simple enough. For example, in geometry, the primitive notions are points, lines, and planes.

2. **Definitions**, or words we say mean stuff said in terms of primitive notions or other definitions. These are precise and are typically motivated by their usefulness. For example, we defined "parallel" because it was useful to have one word to describe when two lines don't intersect (same with the word "intersect"). Yes, we could describe the same situation with only the primitive notions, but it's too many words.

[1] But wait, isn't math about being precise and wouldn't we prefer to have things a bit more clear cut? The answer is, well, we have a limited amount of words, and if we try to define everything exactly we end up running in circles. So we choose simply things that we can get away with not defining–because it's super relatable or something–so we can define the rest rigorously.

3. **Axioms**, or stuff that's taken for granted to be true. What is true within one system may not be true in another. This is where our proofs start. The most commonly cited example is Euclid's 5 axioms form the basis for Euclidean geometry, and relaxing some of those axioms makes for entirely different fields of geometry.

From our axioms we can derive **theorems**, or statements that are proved to be assuming our axioms are true. Typically we don't go all the way down to axioms when proving theorems; we use theorems we've already proved.

You might think at this point that math is very contrived and kind of arbitrary. What's the point if we derive our truths from things we just say are true because it makes sense that way?

And you'd be absolutely correct in saying it's contrived. Unlike natural scientists, mathematicians have chosen making certain statements within a system they understand from the ground up rather than making likely statements *about* a system they don't completely understand. And this certainty is powerful and useful, because the systems are often built in a way that parallels the world we live in.

1.2 Sets

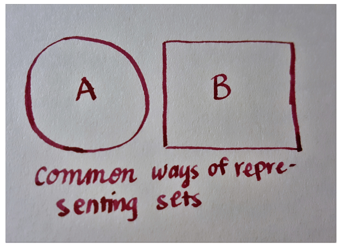
Remember those primitive notions we were talking about in the previous section? The idea of a **set** is one of them.

We can think of a set as a collection of objects. The objects can be anything, including sets themselves! Sets are typically named by capital letters, like so: S. Their contents are enclosed by curly brackets: $\{\}$. A set is determined by the elements inside, and so order and repeated elements don't make sense. (The order in which you list a set's elements isn't going to change whether or not a certain element is in that set, and two of the same thing inside a set won't change whether or not that thing is inside.)

For example $C = \{$Africa, Antartica, Asia, Australia, Europe, North America, South America$\}$ is the set of all the continents on Earth. This set is the same one as $C_1 = \{$Asia, Australia, Europe, Antartica, Africa, North America, South America$\}$.

If an element x is in set A, we can write $x \in A$. So going back to our set of continents A we can write Asia $\in C$. This is read "Asia is in C". If an element y is not in set A we can write $y \notin A$, kind of similarly to $a \neq b$. A example using the set of continents would be USA $\notin A$.

Sometimes we use some type of closed shape to represent a set. Circles and rectangles are most often used because they're pretty easy to draw.



We can write out the elements of a set inside the set, too. This is C, the set of continents from above.

Of course we can't do this with infinite sets but in either case we can think of all the elements as being in the shape.

1.2.1 Cardinality, or Size

It's pretty natural to associate with a set the size, or **cardinality**, of it. Sets can be infinite or finite or have nothing in them at all. There are 7 continents in the world. The number of breeds recognized by the American Kennel Association is some finite number, but the set of integers is infinite. There is exactly one set without anything in it and we have a special name and a special symbol for it—the **empty set**, or $\varnothing$. We'll get back to that in just a little bit.

1.2.2 Subsets and Proper Subsets

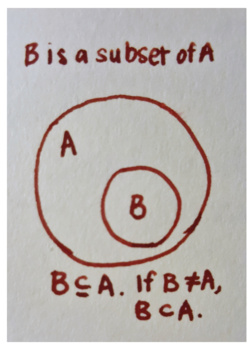
Say I have a paper bag. Then I ask you to choose anything that you'd like from this bag; you can choose everything at all, or you can even choose nothing if you wish. We can call the things in that I had originally in the bag set A and the things you chose set B. We say that B is a **subset** of A because it is made entirely of elements from A. Notation-wise, this is written as $B \subseteq A$.

If not all the items in the bag were enticing enough for you to take them, then we'd have $B \neq A$ and we call B a **proper subset** of A. We signify this with $B \subset A$. (Note the similarity to $\leq$ and $<$).

If none of the items suited you and you picked none of them, then B is the **empty set**, or $\varnothing$. The empty set is a subset of every subset (and a proper subset of everything other than itself).

Even if you picked everything, that's still a subset of A because every element in B is in A.

We have a Venn diagram for this situation too! As you can see every element inside B is also inside A, which is the definition of a subset.



1.2.3 Unions, Intersections, and Complements

New scenario: you are 7 again and you and your sister are collecting Yu-Gi-Oh cards. Since you're a serious collector you only care about cards that are different from each other. So collections are represented by its set of unique cards. [2] Because you're curious you ask your sister to compare collections and see which cards you have in common. If your cards are represented by A and her cards are represented by B, what you're looking at would be the **intersection**—the set of unique cards the two of you have in common—of the two sets, denoted by $A \cap B$.

[2] For the fans: It doesn't matter how many "Exodia the Forbidden One" cards you have, it only matters that you have a complete set of all 5: the legs, arms, and main body.

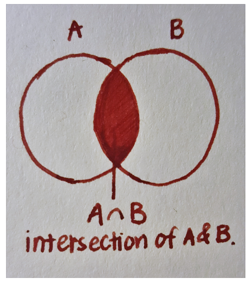


Figure 1.1: The intersection of A and B ($A \cap B$) is a set that has elements in both A and B.

If there are no cards in common, or if $A \cap B = \varnothing$, then A and B are said to be **disjoint**.

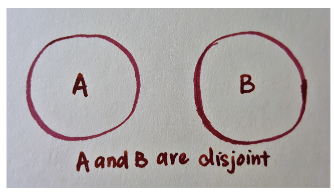


Figure 1.2: A and B are disjoint because they have no elements in common.

Your sister, being younger and purer than you are, proposes to merge the two collections to make a family collection. This would be the union of your collections, or the set of all unique Yu-Gi-Oh cards in the combination of your cards, denoted $A \cup B$. So if

$$A = \{\text{Kuriboh, Summoned Skull, Monster Reborn}\}$$

and if

$$B = \{\text{Blue Eyes Ultimate Dragon, Dark Magician, Summoned Skull, Monster Reborn}\},$$

then

$$A \cup B = \{\text{Kuriboh, Summoned Skull, Monster Reborn, Blue Eyes Ultimate Dragon, Dark Magician}\}$$

or in Venn diagram form,

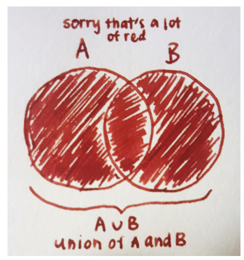


Figure 1.3: The union of sets A and B ($A \cup B$) is the set that contains elements in either A or B.

But since you're a big and mean 7-year-old you refuse and you decide to show her all the different cards you've collected that she has not. This is the **relative complement** of B in A, denoted $A \backslash B$, or $A - B$. More formally, this is all the elements in A that are not in B. Taking A and B to be the same as those in the previous paragraph, we have, in an anticlimactic revelation,

$$A \backslash B = A - B = \{\text{Left Arm of the Forbidden One}\}.$$

A Venn diagram representation of this would be

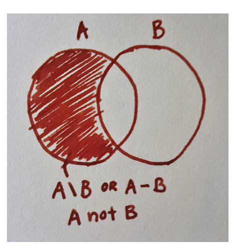


Figure 1.4: The relative complement of B in A ($A \backslash B$ or $A - B$) is set of the elements in A that are not in B.

But then she makes a pouty face and points out that $B\backslash A$ is bigger than $A\backslash B$, since

$$B\backslash A = \{\text{Right Leg of the Forbidden One, Left Leg of the Forbidden One}\}.$$

1.2.4 Wait, so what's the difference between $\in$ and $\subseteq$?? an Example to Tie Everything Together

Let's try to clarify with an example. Say $A = \{1, 2, 3, 4, 5\}$ and $B = \{\{1, 2, 3\}, \{4, 5\}, 6\}$. So what are the elements in each of these sets? A has 6 elements—1, 2, 3, 4, and 5. B, however, has 3 elements, 2 of which are sets—the elements are $\{1, 2, 3\}, \{4, 5\}$, and 6. So we can say that 1 is an element of A, or 1 is in A. So $1 \in A$. However we can't say that 1 is an element of B, because 1 is one of B's 3 elements. So $1 \notin B$.

So in the same vein, we can see that $\{1, 2, 3\} \in B$ and $\{1, 2, 3\} \notin A$. However 1, 2, and 3 are all elements of A so a set with those elements would be a subset of A. So $\{1, 2, 3\} \subseteq A$. But $\{1, 2, 3\} \nsubseteq B$, because B does not contain the elements 1, 2, and 3.

Now I'm just going to list a bunch of true statements based on our definitions two paragraphs ago of A and B. (I don't want to bore you with too much explanation and I feel like I've provided sufficient wordiness, but if you don't why any of these are true don't be afraid to ask someone.)

For reference,

$$A = \{1, 2, 3, 4, 5\}$$

and

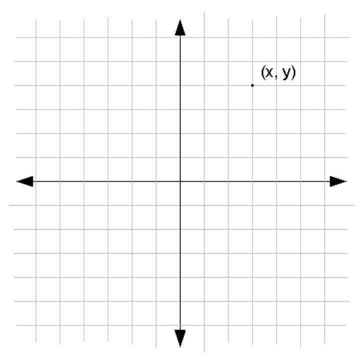
$$B = \{\{1, 2, 3\}, \{4, 5\}, 6\}$$

- $\{4, 5\} \in B$
- $\{4, 5\} \subseteq A$
- $\{3, 4, 5\} \subseteq A$
- $\{3, 4, 5\} \notin B$
- $\{3, 4, 5\} \nsubseteq B$
- $\{\{1, 2, 3\}\} \subseteq B$
- $\{\{1, 2, 3\}, 6\} \subseteq B$
- $\{1, 2, 3, 4, 5\} \subseteq A$
- A and B are disjoint because $A \cap B = \varnothing$ so

- $A - B = A$
- $B - A = B$

1.2.5 Cartesian Products

Remember this from elementary school (and middle school, and high school)?



We usually call this the coordinate plane, but it's also called the *Cartesian plane* sometimes. This is because René Descartes invented it.

So how does the Cartesian plane relate to Cartesian products? The answer lies in how does the plane is constructed. Basically if we're given a line we can describe any point on it by putting a real number line on it. In this scheme, every real number represents an unique point and every point is represented by a unique real number, i.e. the entire line can be represented by exactly the entire set of real numbers.

Descartes's insight was that, in a plane, if we put two number lines perpendicular to each other, every pair of real numbers is represents a unique point on the plane and every point on the plane is represented by a unique pair of real numbers, or the entire plane is represented by exactly the entire set of pairs of real numbers.

Which brings us to the definition of a Cartesian product—given sets A and B, the **Cartesian product** of A and B, denoted $A \times B$ is the set of all possible pairs (a, b) where $a \in A$ and $b \in B$. $\mathbb{R}$ is the set of all real numbers, so in the Cartesian coordinate system, we use $\mathbb{R} \times \mathbb{R}$ as the set of possible coordinates.

As a smaller example, let $A = \{1, 2, 3, 4\}$ and let $B = \{5, 6, 7\}$. Then

$$\begin{aligned} A \times B = \{&(1,5),(2,5),(3,5),(4,5) \\ &(1,6),(2,6),(3,6),(4,6) \\ &(1,7),(2,7),(3,7),(4,7)\} \end{aligned}$$

1.2.6 Important Sets

There are a few important sets of numbers that have special symbols.

- The set of **natural numbers** is denoted by $\mathbb{N}$ and (in this class) contains 0 and the positive integers (or whole numbers). In some circles $\mathbb{N}$ does not include 0.
- The set of **integers** is denoted by $\mathbb{Z}$ and contains 0, the positive whole numbers, and the negative whole numbers.
- The set of **rational numbers** is denoted by $\mathbb{Q}$ (for **Q**uotient) because every rational number q can be written as a quotient of two integers a, b such that $q = a/b$ $(b \neq 0)$.
- The set of **real numbers** is denoted by $\mathbb{R}$. Recall that real numbers include the rationals and the irrationals.
- The set of **complex numbers** is denoted by $\mathbb{C}$. Complex numbers are of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.

Notice that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

1.3 Sums and Products

Summation notation is pretty challenging to get used to for everybody, so just because this symbol: $\sum$ scares you now doesn't mean it'll scare you after a healthy amount of practice! Okay it's probably not challenging to *everyone*, but I confess that not too long ago whenever I saw $\sum$ I kinda freaked out and tried to pretend that everything was ok even though it was not and eventually I got so freaked out and tried to pretend so hard I just couldn't do the problem. But I'm okay now, because I've gotten lots of practice!! If you're still in the phase where it freaks you out, just think about what it really means and rewrite it in a way that makes sense to you.

Which brings us to, what *does*

$$\sum_{i=0}^{n} f(i)$$

mean? It means for every integer (represented by i) from 0 to n (inclusive, which means $0 \leq i \leq n$) we evaluate $f(i)$—so we would have $f(0), f(1), \cdots, f(n)$ and then we add everything up. So now we have

$$\sum_{i=0}^{n} f(i) = f(0) + f(1) + \cdots + f(n)$$

Another way to think of this that we have this format that we're going to apply to integers in a range and then we add everything up after! The "format" is just all the stuff after the sum sign. So now let's do an example!

Example. $\sum_{i=0}^{5} i^2$

Step 1: Break it down. What's $f(i)$, or what are we doing to every integer? We see that we're squaring the integer, or, $f(i) = i^2$.

Step 2: Apply the format and add it all up. So we have

$$\begin{aligned}\sum_{i=0}^{5} i^2 &= 0^2 + 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \\ &= 55\end{aligned}$$

Products work the same way except this time you're multiplying stuff together instead of adding things together! And they use this symbol $\prod$. So

$$\prod_{i=0}^{n} f(i) = f(0) \times f(1) \times f(2) \times \cdots \times f(n)$$

1.4 Quantifiers

In mathematics we like to say stuff like "for all <insert variable [3]> $\in$ <set> there exists <something something>." A familiar one: for all $a \in \mathbb{Q}$ there exist integers p, q such that $a = \frac{p}{q}$.

[3]There are a lot of conventions associated with which variables are which. For example, $a, b, c, d, (e, f)$ are coefficients, and sometimes when there are too many we use $a_0, a_1, \cdots a_n$. We use i, j, k for indexes (think summation notation and iteration) and unit vectors, f, g, h for functions, p for a prime number or probability, m, n for integers, $\alpha, \beta, \gamma, \theta, \varphi$ for angles. There are more but I can't think of any more that we commonly use in this class. (Okay, we don't really use much of the angle stuff, but c'mon. Greek letters are fun to write.) These aren't hard guidelines, but associating meanings to letters will help you read proofs.

Mathematicians are lazy, so they made up symbols for basically everything in that statement. We went over $\in$ in the sets section, and we note we can write "integers p, q" as $p, q \in \mathbb{Z}$. So we're left with "for all", "there exist(s)", and "such that". For **A**ll is upside-down "A": $\forall$, there **E**xist(s) is a backwards "E": $\exists$, and math people are soo lazy that we just decide to omit the "such that" (or we abbreviate it s.t.). So our statement can be written

$$\forall a \in \mathbb{Q} \ \exists p, q \in \mathbb{Z} \ a = \frac{p}{q}$$

I'm personally not a huge fan, but for these, I find it easiest to translate it back into English.