# Note on linearity of expectation

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Suppose $X$ and $Y$ are discrete random variables over the sample space, $S$. ($X : S \to [0,1]$, $Y : S \to [0,1]$).

Show: $\mathbb{E}[X] + \mathbb{E}[Y] = \mathbb{E}[X+Y]$.

Step 1: Show $\mathbb{E}[X] = \sum_{s \in S} X(s)\, p(s)$.

probability that s is the outcome of the experiment

Value of X when s is the outcome of the experiment

Suppose that the distinct values of $X$ are $x_1, x_2, \dots$

Let $S_i$ be the event that $X$ equals $x_i$. ($S_i = \{s \in S : X(s) = x_i\}$)

set of all outcomes for which random variable $X$ takes on the value $x_i$

Sum over all distinct values of $X$

Short hand for $\{s \in S : X(s) = x_i\}$

$$\mathbb{E}[X] = \sum_i x_i \Pr\{X = x_i\}$$ Def of expected value for discrete random variable

$$= \sum_i x_i \Pr(S_i)$$ Def of $S_i$

$$= \sum_i x_i \sum_{s \in S_i} p(s)$$

disjoint events

p(s) is the symbol for the probability of s being the outcome

$$\Pr(S_i) = \Pr\Big(\bigcup_{s \in S_i} \{s\}\Big) = \sum_{s \in S_i} \Pr(\{s\}) = \sum_{s \in S_i} p(s)$$

We can express event $S_i$ as a union of disjoint events $\{s\}$, $s \in S_i$

$$= \sum_i \sum_{s \in S_i} x_i\, p(s)$$ rearrange sums

$$= \sum_i \sum_{s \in S_i} X(s)\, p(s)$$ $X(s) = x_i$, the value of $X$ for outcome s

$$= \sum_{s \in S_1} X(s)\, p(s) + \sum_{s \in S_2} X(s)\, p(s) + \cdots$$

Because $x_1, x_2, \ldots$ are distinct values of $X$,

the associated events $S_1, S_2, \ldots$ are disjoint whose union

is the entire sample space, $S$.

Thus, $S = \bigcup_{i=1} S_i$, where $S_i \cap S_j = \emptyset$ for any $i \neq j$.

Continuing on,

summing over all outcomes associated with event $S_1$

summing over all outcomes associated with event $S_2$, etc.

$$\mathbb{E}[X] = \sum_{s \in S_1} X(s)p(s) + \sum_{s \in S_2} X(s)p(s) + \cdots$$

$$= \sum_{s \in \{S_1 \cup S_2 \cup \ldots\}} X(s)p(s)$$

summing over all outcomes accoiated with every event

$$= \sum_{s \in S} X(s)p(s).$$

summing over all outcomes in the sample space

<u>Ref</u>: Ross, <u>A First Course in Probability</u>, Proposition 9.1, p. 165, 8th edition.

<u>Step 2</u>: Show $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.

$$\mathbb{E}[X+Y] \overset{\text{Step 1}}{=} \sum_{s \in S} \left(X(s) + Y(s)\right) p(s)$$

$$= \sum_{s \in S} X(s)p(s) + \sum_{s \in S} Y(s)p(s)$$

$$\overset{\text{Step 1 applied twice}}{=} \mathbb{E}[X] + \mathbb{E}[Y].$$

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<u>Ref</u>: Ross, <u>A First Course in Probability</u>, Corollary 9.2, p. 166, 8th edition.