Lecture 10: Error Correcting Codes

WCat Can You 2o Cith A Noisy ChahDel?

Sema-Five?

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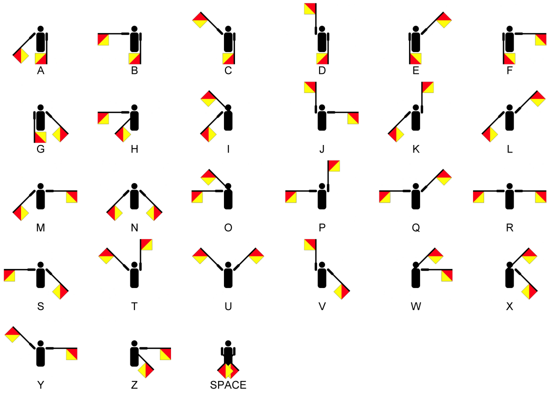

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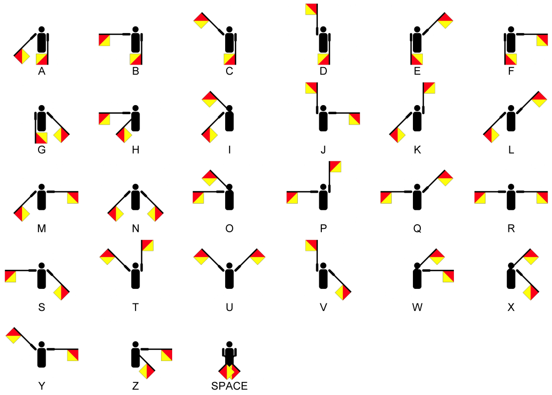

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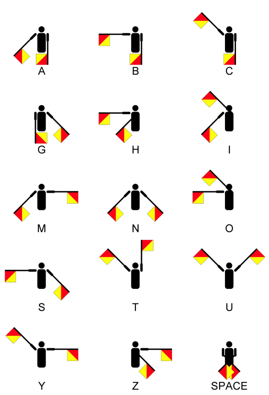

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Less silly: deal with dropped internet packets

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Formally: have message in n parts $m_1, \ldots, m_n$
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Can we do better?

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Interpolate poly $p(x)$ st $p(i) = m_i$ for $1 \leq i \leq n$
Send $p(1)$, $p(2)$, ..., $p(n + k)$

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Send $(p(1), p(2), p(3), p(4), p(5)) = (4, 0, 5, 5, 0)$

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Evaluate for message: $(p'(1), p'(2), p'(3)) = (4, 0, 5)$

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Note: works for *any* padding by k packets

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Don't know which message originally sent!

Relaaaaax

Take a 4 minute break!

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Today's Discussion Question:
What's your strangest family tradition?

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- For any $n+k$ points, at least n uncorrupted
- Those n define the original polynomial

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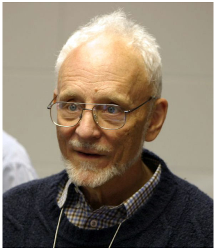
Elwyn Berlekamp

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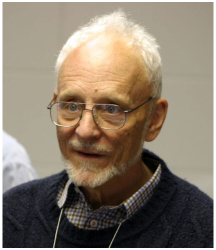
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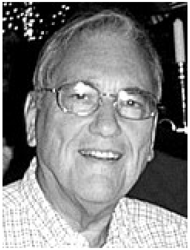
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Berlekamp-Welch Recovery

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Gives $n + 2k$ equations known to be true!
Unknowns are coefficients for $q(x)$ and $e(x)$

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But wait! $b_k = 1$ for any $e_1, \ldots, e_k$!
So $e(x) = x^k + b_{k-1}x^{k-1} + \ldots + b_1 x + b_0$

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But wait! $b_k = 1$ for any $e_1, \dots, e_k$!
So $e(x) = x^k + b_{k-1}x^{k-1} + \dots + b_1x + b_0$

Have $n + k$ unknowns from q, k from e
Matches $n + 2k$ linear eqns of the form $q(i) = r_ie(i)$

Berlekamp-Welch: A Closer Look

What does $q(x)$ look like?
$\deg(p) = n - 1$, $\deg(e) = k$, so $\deg(q) = n + k - 1$
$q(x) = a_{n+k-1}x^{n+k-1} + ... + a_1 x + a_0$

What does $e(x)$ look like?
$e(x) = (x - e_1)(x - e_2)...(x - e_k)$, so degree k
$e(x) = b_k x^k + ... + b_1 x + b_0$

But wait! $b_k = 1$ for any $e_1, ..., e_k$!
So $e(x) = x^k + b_{k-1}x^{k-1} + ... + b_1 x + b_0$

Have $n + k$ unknowns from q, k from e
Matches $n + 2k$ linear eqns of the form $q(i) = r_i e(i)$

Linear Algebra: can find q, e, so have $p(x) = \frac{q(x)}{e(x)}$

Berlekamp-Welch: Example

Want to send length 2 message, have 1 corruption
Receive messages $(1,3)$, $(2,1)$, $(3,4)$, $(4,0)$ mod 7

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Eq 3: $q(3) = r_3e(3)$, so $9a_2 + 3a_1 + a_0 = 4(3 + b_0)$

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Eq 4: $q(4) = r_4e(4)$, so $16a_2 + 4a_1 + a_0 = 0(4 + b_0)$

Berlekamp-Welch: Example

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Eq 3: $q(3) = r_3e(3)$, so $9a_2 + 3a_1 + a_0 = 4(3 + b_0)$

Eq 4: $q(4) = r_4e(4)$, so $16a_2 + 4a_1 + a_0 = 0(4 + b_0)$

Note: all eqns modulo 7, so can shrink some nums

(Berlekamp-Welch: Example): Continued

Simplify equations mod 7, move all variables to left:

$a_2 + a_1 + a_0 - 3b_0 = 3$

$4a_2 + 2a_1 + a_0 - b_0 = 2$

$2a_2 + 3a_1 + a_0 - 4b_0 = 5$

$2a_2 + 4a_1 + a_0 = 0$

(Berlekamp-Welch: Example): Continued

Simplify equations mod 7, move all variables to left:

$a_2 + a_1 + a_0 - 3b_0 = 3$
$4a_2 + 2a_1 + a_0 - b_0 = 2$
$2a_2 + 3a_1 + a_0 - 4b_0 = 5$
$2a_2 + 4a_1 + a_0 = 0$

Can use Gaussian Elimination (mod 7) to solve

(Berlekamp-Welch: Example): Continued

Simplify equations mod 7, move all variables to left:

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$2a_2 + 4a_1 + a_0 = 0$

Can use Gaussian Elimination (mod 7) to solve

Here, $a_2 = 3$, $a_1 = 6$, $a_0 = 5$, $b_0 = 6$
So $q(x) = 3x^2 + 6x + 5$, $e(x) = x + 6$

(Berlekamp-Welch: Example): Continued

Simplify equations mod 7, move all variables to left:

$a_2 + a_1 + a_0 - 3b_0 = 3$
$4a_2 + 2a_1 + a_0 - b_0 = 2$
$2a_2 + 3a_1 + a_0 - 4b_0 = 5$
$2a_2 + 4a_1 + a_0 = 0$

Can use Gaussian Elimination (mod 7) to solve

Here, $a_2 = 3$, $a_1 = 6$, $a_0 = 5$, $b_0 = 6$
So $q(x) = 3x^2 + 6x + 5$, $e(x) = x + 6$

Do poly long division mod 7 to get $p(x) = 3x + 2$
Original messages: $p(1) = 5$, $p(2) = 1$

Fin

Next time: countability!