Lecture 11: Countability

Infinity is weeeeeeird

What Is "Same Size"?

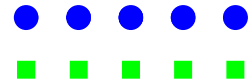
Consider two sets:
$\{1, 2, 3, 4\}$
$\{0, 1, 2, 3, 4\}$

Are these the same size?
No! Second set has an extra element!

What about:
$\mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\}$
$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$

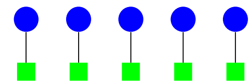
$\mathbb{N}$ has an extra element...but both are infinite?
Is $\infty + 1 = \infty$?
????
Need different way to think about "size"

Finite Example



Are there same number of circles and squares?
How do we know? I can't count to 5...

Idea: Draw lines between squares and circles



Only possible if same number of squares and circles!

How to generalize to infinite sets?

Bijections and Size

Idea: sets "same size" if $\exists$ bijection between them

Does this make sense for finite sets?

Suppose have bijection $b : \{1, 2, 3\} \to S$
How many elements in S?
$S = \{b(1), b(2), b(3)\}$, so 3 elements as well!

Bijections capture the "same num of elts" idea
But also makes sense for infinite sets!

Same Infinities

Claim: $|\mathbb{N}| = |\mathbb{Z}^+|$[1]

How can we prove this?
Need a bijection!

Claim: $f(x) = x + 1$ is bijection $\mathbb{N} \to \mathbb{Z}^+$
Why? Has inverse $f^{-1}(y) = y - 1$

But what about $f(x) = x$? Not onto!
Don't need all functions bijective! Only need one.

Adding one elt to infinite set doesn't seem to change size...what if we added more?

[1] Here $|S|$ means the cardinality or "size" of S

More Infinities

Claim: $|\mathbb{N}| = |\mathbb{Z}|$

How can we map from $\mathbb{N}$ to $\mathbb{Z}$?
$0 \to 0$
$1 \to 1$
$2 \to -1$
$3 \to 2$
$4 \to -2$
...

Take $f(x) = \begin{cases} \frac{x+1}{2} & x \text{ is odd} \\ -\frac{x}{2} & x \text{ is even} \end{cases}$

Inverse is $f^{-1}(y) = \begin{cases} 2y - 1 & y > 0 \\ -2y & y \leq 0 \end{cases}$

Bijection Alternatives

Explicitly stating a bijection can be a pain...
What alternatives do we have?

To prove $|S| = |\mathbb{N}|$, can give *enumeration* of S:
List "1st" elt of S, then "2nd", then "3rd", etc.
Need to eventually hit every element

Ex: For $\mathbb{Z}$, can enumerate as
$0, 1, -1, 2, -2, 3, -3, \ldots$

Careful — need finite position for any element!
Ex: $0, 1, 2, \ldots, -1, -2, -3, \ldots$ *not* valid for $\mathbb{Z}$

Enumeration Example

Definition: $\{0,1\}^*$ is set of finite bit strings

Theorem: $|\{0,1\}^*| = |\mathbb{N}|$

Could give bijection, but lots of words

Instead, enumerate:
$\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, \ldots$

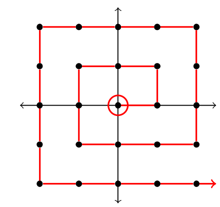
Any string *with finite length* hit eventually!

Have Some More Enumeration

Theorem: $|\mathbb{Z} \times \mathbb{Z}| = |\mathbb{N}|$

Should be surprising — seems like many more pairs!

Proof by picture:



Gives an enumeration of $\mathbb{Z} \times \mathbb{Z}$!

Be Rational!

How does $|\mathbb{Q}|$ compare to $|\mathbb{Z} \times \mathbb{Z}|$?

Can create function $f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$ as follows:
If $q = \frac{a}{b}$ in lowest terms, $f(q) = (a, b)$
$f(2) = (2, 1)$, $f(0.25) = (1, 4)$, $f(0.\overline{66}) = (2, 3)$, etc.

Is f a bijection?
No! Not onto (eg $(1, 0)$, $(-1, -1)$, $(2, 4)$, ...)

But notice: is one-to-one
Can we conclude anything from this?

What Is An Outjection?

Cantor-Schröder-Bernstein Theorem
If $\exists$ injections $f : A \to B$ and $g : B \to A$, $\exists$ bijection

Proof in Bonus Lecture tomorrow!

What does this mean to us?
Can say $|A| \leq |B|$ if $\exists$ injection $f : A \to B$
If $|A| \leq |B|$ and $|B| \leq |A|$, CSB says $|A| = |B|$!

Note: Have inject $A \to B$ iff have surject $B \to A$
So surjection $B \to A$ means $|B| \geq |A|$!

Back To $\mathbb{Q}$

Previously: found injection $\mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$
Hence, $|\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{Z}| = |\mathbb{N}|$

Notice, have injection $\mathbb{N} \to \mathbb{Q}$ by "inclusion"
So $|\mathbb{N}| \leq |\mathbb{Q}|$

Thus $|\mathbb{Q}| = |\mathbb{N}|$!

Brake

Time for a 4-minute break!

Today's Discussion Question:
https://tinyurl.com/70-discussion-q

Countability

Say a set S is *countable* if $|S| \leq |\mathbb{N}|$
So far, all sets we've seen are countable!

Natural question: are all sets countable?
Turns out, no!

Not With That Attitude You Cant-or

Def: Let $\{0,1\}^\infty$ be set of *infinite length* bit strings
Theorem: $|\{0,1\}^\infty| > |\mathbb{N}|$

Proof:
Suppose for contra $\exists$ onto fn $o : \mathbb{N} \to \{0,1\}^\infty$

n	$o(n)$
0	0 0 0 0 0 ...
1	1 0 1 0 1 ...
2	1 1 1 0 1 ...
3	0 1 0 0 0 ...
$\vdots$	$\vdots$

Consider $s = 1101...$

$s \neq o(n)$ for all n!

Method known as *Cantor Diagonalization*

I Cant-or Think Of A Better Pun

Theorem: $|\mathbb{R}| > |\mathbb{N}|$
Will in fact prove $|[0,1]| > |\mathbb{N}|$

"Proof":
Suppose for contra $\exists$ onto fn $o : \mathbb{N} \to [0,1]$

n	$o(n)$
0	.0 9 9 9 9 ...
1	.1 9 2 9 3 ...
2	.0 0 9 0 0 ...
3	.2 3 5 9 6 ...
$\vdots$	$\vdots$

Consider $r = .1000...$

$r \neq o(n)$ for all n!

So we've proved $|[0,1]| > |\mathbb{N}|$ ...or have we?

Oops

Slight subtlety with $\mathbb{R}$:
Decimal expansion not always unique!

Eg, $.09999... = .10000...$

$+1$ to daig ensures different decimal expansion
Not necessarily different number!

In our picture, $o(0) = 0.999... = .1000... = r$

Easily recoverable: just do $+2$ instead of $+1$

Moral: be careful when claiming $r \neq o(n)$!

Not Recoverable

"Theorem": $|\mathbb{Q}| > |\mathbb{N}|$

"Proof":
Suppose for contra $\exists$ onto fn $o : \mathbb{N} \to \mathbb{Q} \cap [0,1]$

n	$o(n)$
0	.1 9 1 9 1 ...
1	.5 9 2 2 2 ...
2	.0 0 2 0 0 ...
3	.6 9 5 9 5 ...
$\vdots$	$\vdots$

Consider $q = .3141...$

$q \neq o(n)$ for all n!

So we've proved $|\mathbb{Q} \cap [0,1]| > |\mathbb{N}|$ ...or have we?

Double Oops

How do we know that q is rational? We don't!
In picture, $q = \frac{\pi}{10} \notin \mathbb{Q}$

Doesn't matter that $q \neq o(n)$
Not trying to cover q!

This proof not recoverable — $|\mathbb{Q}| = |\mathbb{N}|$

Moral: make sure construct in required set!

Fin

Next time: computability!