Lecture 11: Countability

Infinity is weeeeeeird

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$\{0, 1, 2, 3, 4\}$

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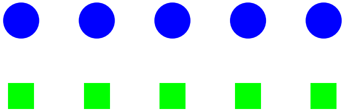
Need different way to think about “size”

Finite Example



Are there same number of circles and squares?

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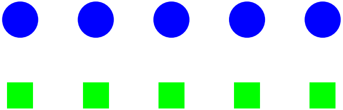

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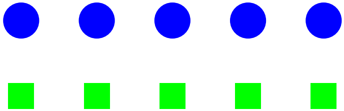

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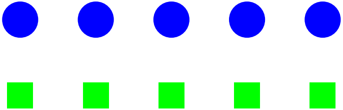

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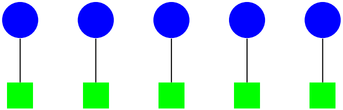

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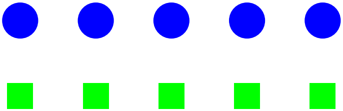

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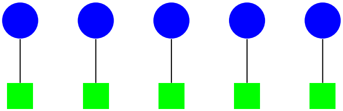

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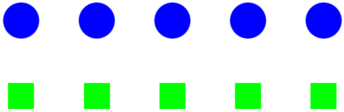

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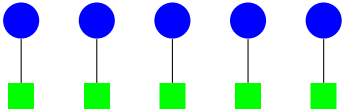

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How to generalize to infinite sets?

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Bijections capture the "same num of elts" idea
But also makes sense for infinite sets!

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Adding one elt to infinite set doesn't seem to change size...what if we added more?

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Take $f(x) = \begin{cases} \frac{x+1}{2} & x \text{ is odd} \\ -\frac{x}{2} & x \text{ is even} \end{cases}$

Inverse is $f^{-1}(y) = \begin{cases} 2y - 1 & y > 0 \\ -2y & y \leq 0 \end{cases}$

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Careful — need finite position for any element!
Ex: $0, 1, 2, \ldots, -1, -2, -3, \ldots$ *not* valid for $\mathbb{Z}$

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Any string *with finite length* hit eventually!

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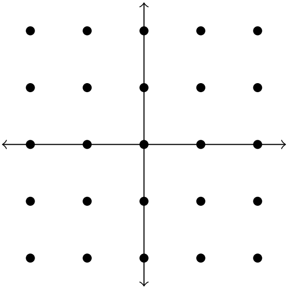

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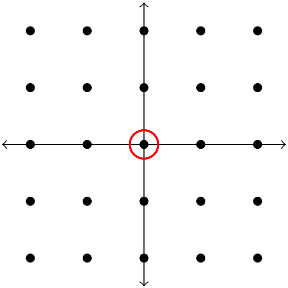

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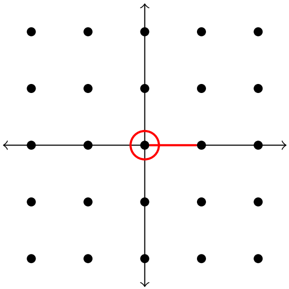

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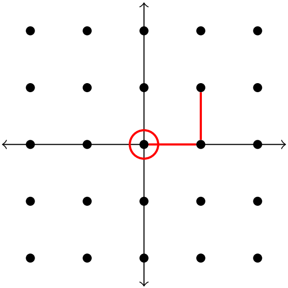

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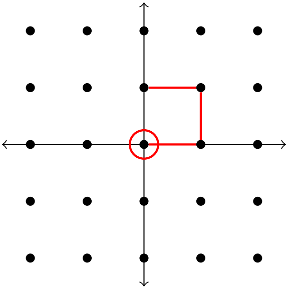

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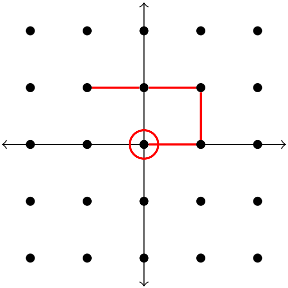

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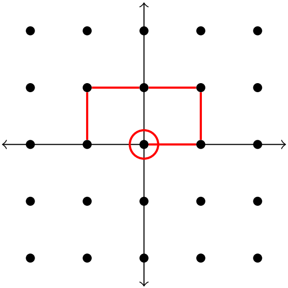

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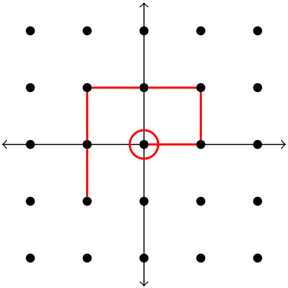

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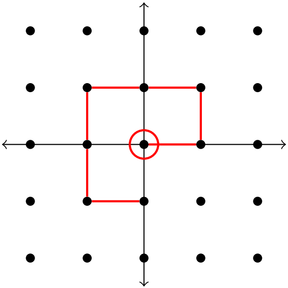

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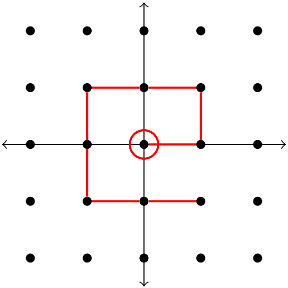

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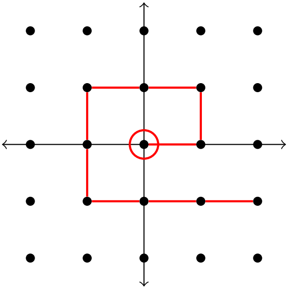

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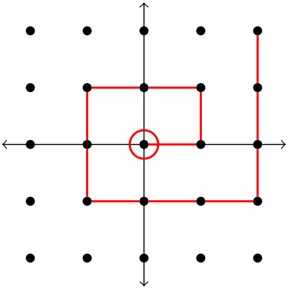

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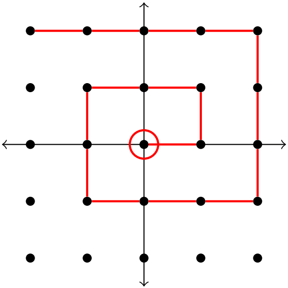

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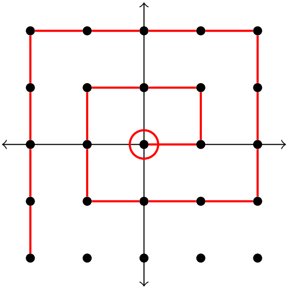

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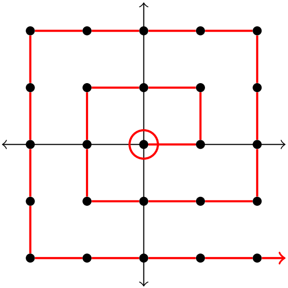

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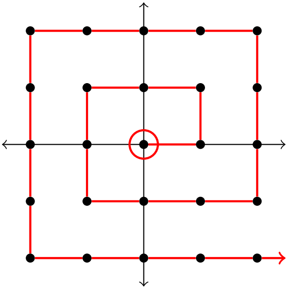

Gives an enumeration of $\mathbb{Z} \times \mathbb{Z}$!

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But notice: is one-to-one
Can we conclude anything from this?

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Cantor-Schröder-Bernstein Theorem

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Proof in Bonus Lecture tomorrow!

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Proof in Bonus Lecture tomorrow!

What does this mean to us?

What Is An Outjection?

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What does this mean to us?
Can say $|A| \leq |B|$ if $\exists$ injection $f\colon A \to B$
If $|A| \leq |B|$ and $|B| \leq |A|$, CSB says $|A| = |B|$!

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Note: Have inject $A \to B$ iff have surject $B \to A$

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Can say $|A| \leq |B|$ if $\exists$ injection $f: A \to B$
If $|A| \leq |B|$ and $|B| \leq |A|$, CSB says $|A| = |B|$!

Note: Have inject $A \to B$ iff have surject $B \to A$
So surjection $B \to A$ means $|B| \geq |A|$!

Back To $\mathbb{Q}$

Previously: found injection $\mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$

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Notice, have injection $\mathbb{N} \to \mathbb{Q}$ by "inclusion"
So $|\mathbb{N}| \leq |\mathbb{Q}|$

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Notice, have injection $\mathbb{N} \to \mathbb{Q}$ by “inclusion”
So $|\mathbb{N}| \leq |\mathbb{Q}|$

Thus $|\mathbb{Q}| = |\mathbb{N}|$!

Brake

Time for a 4-minute break!

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Today's Discussion Question:
https://tinyurl.com/70-discussion-q

Countability

Say a set S is *countable* if $|S| \leq |\mathbb{N}|$

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Say a set S is *countable* if $|S| \leq |\mathbb{N}|$
So far, all sets we've seen are countable!

Natural question: are all sets countable?
Turns out, no!

Not With That Attitude You Cant-or

Def: Let $\{0, 1\}^\infty$ be set of *infinite length* bit strings

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Proof:
Suppose for contra $\exists$ onto fn $o : \mathbb{N} \to \{0,1\}^\infty$

n	$o(n)$
0	0 0 0 0 0 ...
1	1 0 1 0 1 ...
2	1 1 1 0 1 ...
3	0 1 0 0 0 ...
⋮	⋮

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0	0 0 0 0 0 ...
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⋮	⋮

Consider $s = 1$

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Theorem: $|\{0,1\}^\infty| > |\mathbb{N}|$

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Suppose for contra $\exists$ onto fn $o : \mathbb{N} \to \{0,1\}^\infty$

n	$o(n)$
0	0 0 0 0 0 ...
1	1 0 1 0 1 ...
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⋮	⋮

Consider $s = 11$

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Theorem: $|\{0,1\}^\infty| > |\mathbb{N}|$

Proof:
Suppose for contra $\exists$ onto fn $o : \mathbb{N} \to \{0,1\}^\infty$

n	$o(n)$
0	0 0 0 0 0 ...
1	1 0 1 0 1 ...
2	1 1 1 0 1 ...
3	0 1 0 0 0 ...
⋮	⋮

Consider $s = 110$

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Theorem: $|\{0,1\}^\infty| > |\mathbb{N}|$

Proof:
Suppose for contra $\exists$ onto fn $o : \mathbb{N} \to \{0,1\}^\infty$

n	$o(n)$
0	0 0 0 0 0 ...
1	1 0 1 0 1 ...
2	1 1 1 0 1 ...
3	0 1 0 0 0 ...
⋮	⋮

Consider $s = 1101$

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Def: Let $\{0,1\}^\infty$ be set of *infinite length* bit strings
Theorem: $|\{0,1\}^\infty| > |\mathbb{N}|$

Proof:
Suppose for contra $\exists$ onto fn $o : \mathbb{N} \to \{0,1\}^\infty$

n	$o(n)$
0	0 0 0 0 0 …
1	1 0 1 0 1 …
2	1 1 1 0 1 …
3	0 1 0 0 0 …
$\vdots$	$\vdots$

Consider $s = 1101\ldots$

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0	0 0 0 0 0 …
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Consider $s = 1101\ldots$

$s \neq o(n)$ for all n!

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Method known as *Cantor Diagonalization*

I Cant-or Think Of A Better Pun

Theorem: $|\mathbb{R}| > |\mathbb{N}|$
Will in fact prove $|[0,1]| > |\mathbb{N}|$

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0	.0 9 9 9 9 ...
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2	.0 0 9 0 0 ...
3	.2 3 5 9 6 ...
⋮	⋮

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3	.2 3 5 9 6 ...
$\vdots$	$\vdots$

Consider $r = .1$

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Suppose for contra $\exists$ onto fn $o : \mathbb{N} \to [0,1]$

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0	.0 9 9 9 9 ...
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2	.0 0 9 0 0 ...
3	.2 3 5 9 6 ...
$\vdots$	$\vdots$

Consider $r = .10$

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“Proof”:
Suppose for contra $\exists$ onto fn $o : \mathbb{N} \to [0,1]$

n	$o(n)$
0	.0 9 9 9 9 ...
1	.1 9 2 9 3 ...
2	.0 0 9 0 0 ...
3	.2 3 5 9 6 ...
$\vdots$	$\vdots$

Consider $r = .100$

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Theorem: $|\mathbb{R}| > |\mathbb{N}|$
Will in fact prove $|[0,1]| > |\mathbb{N}|$

“Proof”:
Suppose for contra $\exists$ onto fn $o : \mathbb{N} \to [0,1]$

n	$o(n)$
0	.0 9 9 9 9 ...
1	.1 9 2 9 3 ...
2	.0 0 9 0 0 ...
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Consider $r = .1000$

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Will in fact prove $|[0,1]| > |\mathbb{N}|$

"Proof":
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0	.0 9 9 9 9 ...
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Consider $r = .1000...$

$r \neq o(n)$ for all n!

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$\vdots$	$\vdots$

Consider $r = .1000\ldots$

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So we’ve proved $|[0,1]| > |\mathbb{N}|$ ...or have we?

Oops

Slight subtlety with $\mathbb{R}$:
Decimal expansion not always unique!

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In our picture, $o(0) = 0.999... = .1000... = r$

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Easily recoverable: just do $+2$ instead of $+1$

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$+1$ to daig ensures different decimal expansion
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In our picture, $o(0) = 0.999... = .1000... = r$

Easily recoverable: just do $+2$ instead of $+1$

Moral: be careful when claiming $r \neq o(n)$!

Not Recoverable

"Theorem": $|\mathbb{Q}| > |\mathbb{N}|$

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"Proof":
Suppose for contra $\exists$ onto fn $o : \mathbb{N} \to \mathbb{Q} \cap [0, 1]$

n	$o(n)$
0	.1 9 1 9 1 ...
1	.5 9 2 2 2 ...
2	.0 0 2 0 0 ...
3	.6 9 5 9 5 ...
⋮	⋮

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0	.1 9 1 9 1 ...
1	.5 9 2 2 2 ...
2	.0 0 2 0 0 ...
3	.6 9 5 9 5 ...
⋮	⋮

Consider $q = .3$

Not Recoverable

"Theorem": $|\mathbb{Q}| > |\mathbb{N}|$

"Proof":

Suppose for contra $\exists$ onto fn $o : \mathbb{N} \to \mathbb{Q} \cap [0, 1]$

n	$o(n)$
0	.1 9 1 9 1 ...
1	.5 9 2 2 2 ...
2	.0 0 2 0 0 ...
3	.6 9 5 9 5 ...
$\vdots$	$\vdots$

Consider $q = .31$

Not Recoverable

"Theorem": $|\mathbb{Q}| > |\mathbb{N}|$

"Proof":
Suppose for contra $\exists$ onto fn $o : \mathbb{N} \to \mathbb{Q} \cap [0,1]$

n	$o(n)$
0	.1 9 1 9 1 ...
1	.5 9 2 2 2 ...
2	.0 0 2 0 0 ...
3	.6 9 5 9 5 ...
⋮	⋮

Consider $q = .314$

Not Recoverable

"Theorem": $|\mathbb{Q}| > |\mathbb{N}|$

"Proof":
Suppose for contra $\exists$ onto fn $o : \mathbb{N} \to \mathbb{Q} \cap [0, 1]$

n	$o(n)$
0	.1 9 1 9 1 ...
1	.5 9 2 2 2 ...
2	.0 0 2 0 0 ...
3	.6 9 5 9 5 ...
⋮	⋮

Consider $q = .3141$

Not Recoverable

"Theorem": $|\mathbb{Q}| > |\mathbb{N}|$

"Proof":
Suppose for contra $\exists$ onto fn $o : \mathbb{N} \to \mathbb{Q} \cap [0, 1]$

n	$o(n)$
0	.1 9 1 9 1 ...
1	.5 9 2 2 2 ...
2	.0 0 2 0 0 ...
3	.6 9 5 9 5 ...
$\vdots$	$\vdots$

Consider $q = .3141...$

Not Recoverable

"Theorem": $|\mathbb{Q}| > |\mathbb{N}|$

"Proof":
Suppose for contra $\exists$ onto fn $o : \mathbb{N} \to \mathbb{Q} \cap [0,1]$

n	$o(n)$
0	.1 9 1 9 1 ...
1	.5 9 2 2 2 ...
2	.0 0 2 0 0 ...
3	.6 9 5 9 5 ...
$\vdots$	$\vdots$

Consider $q = .3141...$

$q \neq o(n)$ for all n!

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"Proof":
Suppose for contra $\exists$ onto fn $o : \mathbb{N} \to \mathbb{Q} \cap [0, 1]$

n	$o(n)$
0	.1 9 1 9 1 ...
1	.5 9 2 2 2 ...
2	.0 0 2 0 0 ...
3	.6 9 5 9 5 ...
⋮	⋮

Consider $q = .3141...$

$q \neq o(n)$ for all n!

So we've proved $|\mathbb{Q} \cap [0, 1]| > |\mathbb{N}|$

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"Theorem": $|\mathbb{Q}| > |\mathbb{N}|$

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Suppose for contra $\exists$ onto fn $o : \mathbb{N} \to \mathbb{Q} \cap [0, 1]$

n	$o(n)$
0	.1 9 1 9 1 ...
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2	.0 0 2 0 0 ...
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$\vdots$	$\vdots$

Consider $q = .3141...$

$q \neq o(n)$ for all n!

So we've proved $|\mathbb{Q} \cap [0, 1]| > |\mathbb{N}|$ ...or have we?

Double Oops

How do we know that q is rational?

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Not trying to cover q!

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This proof not recoverable — $|\mathbb{Q}| = |\mathbb{N}|$

Moral: make sure construct in required set!

Fin

Next time: computability!