Lecture 11: Self-Reference and Uncomputability

Self-Referential Subtitles Are Best Subtitles

Liar's Paradox

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- Is it?

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Set theory solution: make sure S not definable

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Set theory solution: make sure S not definable

In CS, not so easy to avoid!

An Important Aside

Computer programs $\equiv$ binary strings

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```python
import sys

def example_function():
    print("This is an example function")
    print("It does example things")
    sys.crash()
```

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Means we can pass programs as inputs to programs

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Means we can pass programs as inputs to programs
Program can be own input — allows self-reference!

An Impossible Problem

Halting problem: determine if program halts

Formally, want program `TestHalt` such that

- If `P(x)` halts, `TestHalt(P, x)` = True
- If `P(x)` loops, `TestHalt(P, x)` = False

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To prove: assume for contradiction `TestHalt` exists
Use self-reference to defeat `TestHalt`

Turing The Computer Scientist

Assume for contradiction `TestHalt` exists

```
Turing(P):
   if TestHalt(P, P) = True:
       loop infinitely
   else:
       halt
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What does `Turing(Turing)` do?

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What does `Turing(Turing)` do?

Opposite of `TestHalt(Turing, Turing)`

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What does `Turing(Turing)` do?

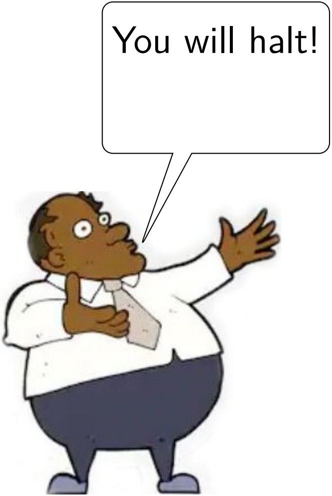
Opposite of `TestHalt(Turing, Turing)`
So `TestHalt` must be wrong there!

Turing The Sassy Teenager

`TestHalt`

`Turing`

Turing The Sassy Teenager



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Turing The Sassy Teenager

You will halt!

I’ll loop instead!

`TestHalt`

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You will halt!

Fine, loop!

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Fine, loop!

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Actually, I'll halt!

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But Wait!

Why can't we just simulate `P(x)` and wait for halt?

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`P(x)` might need 9001!

OK Sure, But...

...maybe `TestHalt` is just contrived?
Don't often care what program does on itself

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Perhaps better: does program halt with no input?

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"Easy" Halting Problem: want `ETH` such that

- If `P()` halts, `ETH(P)` = True
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- If `P()` halts, `ETH(P)` = True
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Claim: "Easy" Halting Problem no easier!

Formally: if `ETH` exists, `TestHalt` does too

Easy My ***

Suppose `ETH` exists, can write `TestHalt`:

```
TestHalt(P, x):
    def P'():
        P(x)
    return ETH(P')
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Contrapos: `TestHalt` doesn't exist $\Longrightarrow$ `ETH` doesn't

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In logic: `ETH` exists $\implies$ `TestHalt` exists
Contrapos: `TestHalt` doesn't exist $\implies$ `ETH` doesn't

Already Know `TestHalt` doesn't exist!

Reductionism

What we just did called a *reduction*

`TestHalt`

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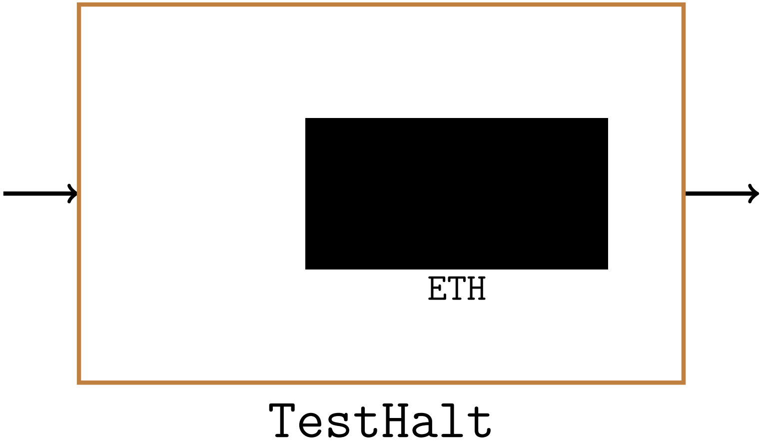

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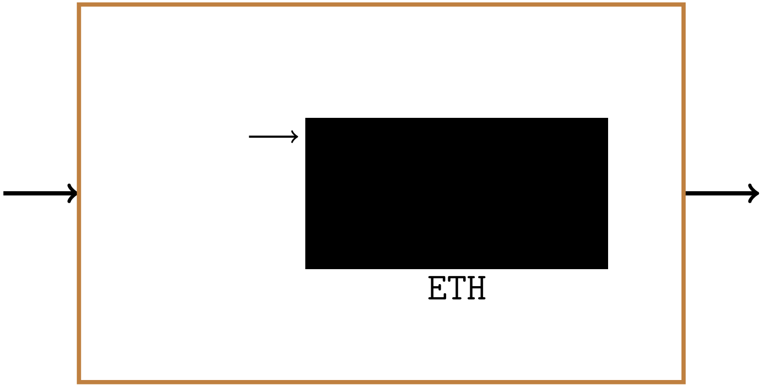

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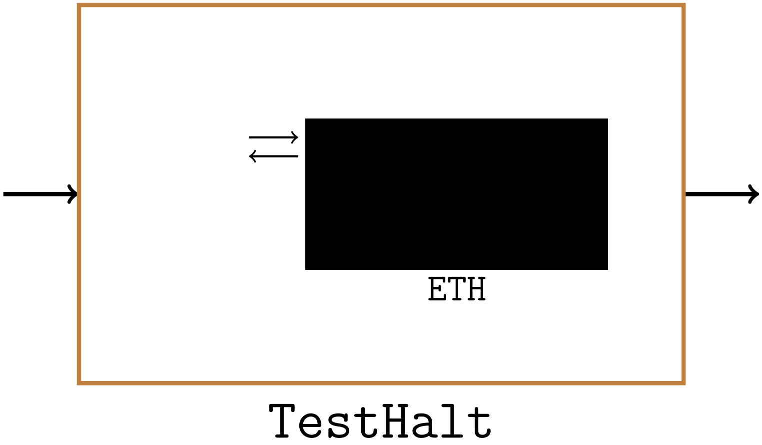

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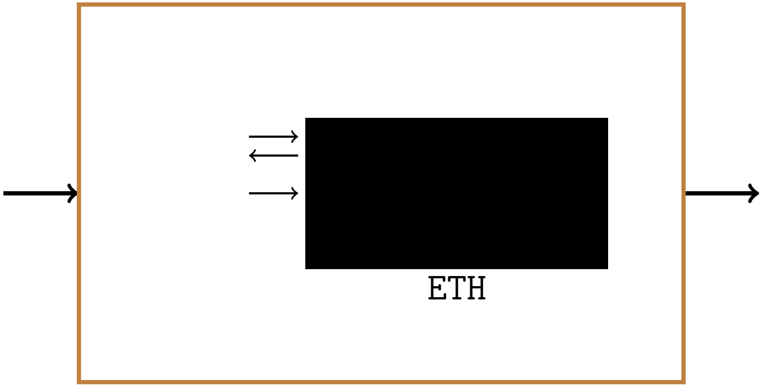

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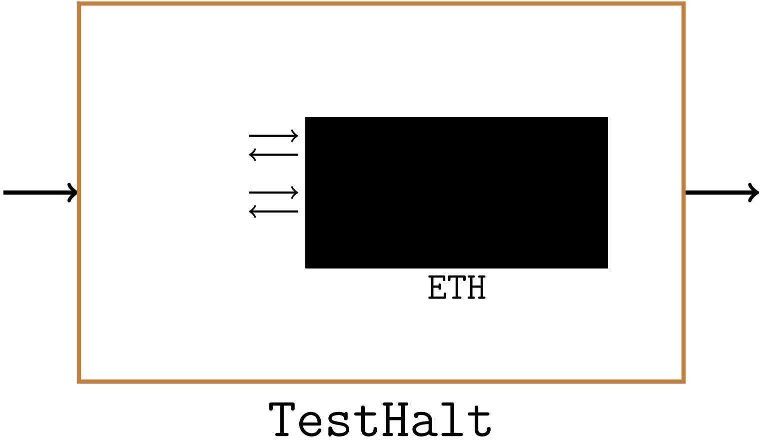

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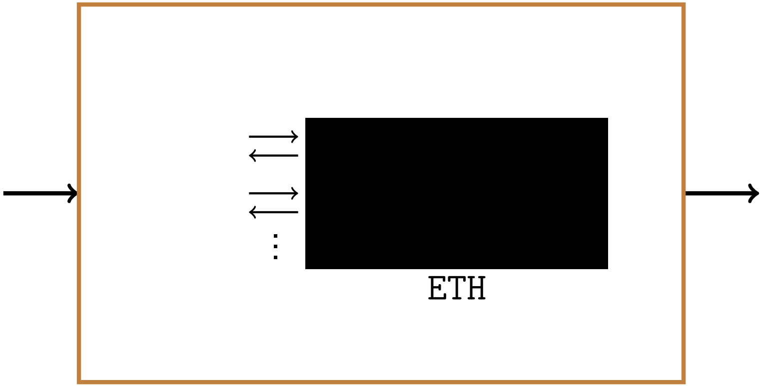

TestHalt

Reduce To The Problem Of Break Time

Time for a 4-minute break!

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Today's Discussion Question:
If you were to write a self-referential discussion question, what would it be?

Recursive Enumerability

What happens if we relax the requirements?

[1]Sometimes called *recognizable*, but that doesn't sound as cool.

Recursive Enumerability

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Problem is *recursively enumerable*[1] if $\exists$ program `P`

- If answer for x is true, `P(x)` outputs true
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Can we find others?

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Allows us to solve Halting Problem — no bueno!

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Note: requires two important assumptions

- Proofs can be checked for correctness
- Only finitely many possible next steps

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Both true in sufficiently formal proof systems!

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Used `TestHalt` to avoid problems if `R` loops!

Give Out Complements

What's so special about the false case?
What happens if we relax the true case instead?

[2] The co- stands for "complement"

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Problem is co-RE[2] if $\exists$ program `P` st

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Ex: the "looping problem" is co-RE

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Can RE problems be co-RE as well?

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RE(EEEEEEEEEE)

Thm: Problem is RE and co-RE iff is computable

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Proof (if):

- Solver satisfies both RE and co-RE

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$\exists$ problems neither RE nor co-RE!
Beyond our scope though :'(

Fin

Next time: counting (with Elizabeth)!