Counting, Part I

CS 70, Summer 2019

Lecture 13, 7/16/19

Goals: Probability

- Lets you **quantify** uncertainty
- Concretely: has **applications** everywhere!
- Hopefully: learn techniques for **reasoning about randomness** and **making better decisions** logically
- Hopefully: provides a **new perspective** on the world

CS 70 Tips

The probability section in CS 70 usually means:

- **One big topic**, rather than many small topics
  - Try your best to stay **up to date**; use OH!
  - Important to be comfortable with the **basics**
- Fewer "proofs," more **computations**
  - Emphasis on **applying tools** and **problem solving**
  - Lectures will be **example-driven**
- **Practice, practice, practice!**

A Familiar Question

How many bit (0 or 1) strings are there of length 3?

Choices, Choices, Choices...

A lunch special lets you choose one appetizer, one entreé, and one drink. There are 6 appetizers, 3 entreés, and 5 drinks. How many different meals could you possibly get?

The First Rule of Counting: Products

If the object you are counting:

- Comes from making k **choices**
- Has n_1 options for the first choice
- Has n_2 options for second, **regardless of the first**
- Has n_3 options for the third, **regardless of the first two**
- ...and so on, until the k-th choice

$\implies$ Count the object using the **product**

$$n_1 \times n_2 \times n_3 \times \ldots \times n_k$$

Anagramming I

How many strings can we make by rearranging "CS70"?

How many strings can we make by rearranging "ILOVECS70" if the numbers "70" must appear together in that order?

Counting Functions

How many functions are there from $\{1, \ldots, n\}$ to $\{1, \ldots, m\}$?

Same setup, but $m \geq n$. How many injective functions are there?

Counting Polynomials

How many degree d polynomials are there modulo p?

If $d \leq p$, how many have no repeating coefficients?

When Order Doesn't Matter: Space Team I

Among its 10 trainees, NASA wants to choose 3 to go to the moon. How many ways can they do this?

When Order Doesn't Matter: Poker I

In poker, each player is dealt 5 cards. A standard deck (no jokers) has 52 cards. How many different hands could you get?

The Second Rule of Counting: Repetitions

Say we use the First Rule–we make k choices.

- Let A be the set of **ordered** objects.
- Let B be the set of **unordered** objects.

If there is an **"m-to-1"** function from A to B:

$\Longrightarrow$ Count A and divide by m to get $|B|$.

Anagramming II

How many strings can we make by rearranging "APPLE"?

How many strings can we make by rearranging "BANANA"?

Binomial Coefficients

How many ways can we...

- pick a set of 2 items out of n total?
- pick a set of 3 items out of n total?
- pick a set of k items out of n total?

Binomial Coefficients

We often use

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

to represent the number of ways to choose k out of n items when order doesn't matter.

We call this quantity **"n choose k"**.
We also sometimes refer to these as "binomial coefficients."

Q: Using this definition, what does 0! equal?

Binomial Coefficients

Using this definition, what does 0! equal?

Should we be surprised that $\binom{n}{k} = \binom{n}{n-k}$?

Anagramming III

How many bit strings can we make by k 1's and $(n-k)$ 0's?

Coincidence?

Is there a relationship between:

1. Length n bit strings with k 1's, and
2. Ways of choosing k items from n when order doesn't matter?

Yes!

Putting It All Together: Space Team II

Among its 10 trainees, NASA wants to choose 3 to go to the moon, and 2 to go to Mars. They also don't want anyone to do both missions. How many ways can they choose teams?

If one member of the moon mission is designated as a captain, how many ways can they choose teams?

Putting It All Together: Poker II

How many 5-card poker hands form a full house (triple + pair)?

How many 5-card poker hands form a straight (consecutive cards), including straight flushes (same suit)?

How many 5-card poker hands form two pairs?

Sampling Without Replacement

How many ways can we sample k items out of n items, **without replacement**, if:

- Order matters?
- Order does not matter?

We were able to use the First and Second rules of counting!

Sampling With Replacement

How many ways can we sample k items out of n total items, **with replacement**, if:

- Order matters?
- Order does not matter?

What can we do when order does not matter?

When Repetitions Aren't Uniform: Splitting Money

Alice, Bob, and Charlie want to split \$6 amongst themselves.

First (naive and difficult) attempt: the "dollar's point of view"

When Repetitions Aren't Uniform: Splitting Money

Second attempt: the "divider" point of view

"Stars and Bars" Application: Sums to k

How many ways can we choose n (not necessarily distinct) non-negative numbers that sum to k?

Food for thought: What if the numbers have to be **positive**?

Summary

- k choices, always the same number of options at choice i regardless of previous outcome $\implies$ **First Rule**
- Order doesn't matter; same number of repetitions for each desired outcome $\implies$ **Second Rule**
- Indistinguishable items split among a fixed number of different buckets $\implies$ **Stars and Bars**

Pick Your Strategy I

You have 12 distinct cards and 3 people. How many ways to:

- Deal to the 3 people in sequence (4 cards each), and the order they received the cards matters?
- Deal to the 3 people in sequence (4 cards each), but order doesn't matter?

Pick Your Strategy II

You have 12 distinct cards and 3 people. How many ways to:

- Deal 3 piles in sequence (4 cards each), and don't distinguish the piles?
- The cards are now indistinguishable. How many ways to deal so that each person receives at least 2 cards?

Pick Your Strategy III

There are n citizens on 5 different committees.
Say $n > 15$, and that each citizen is on at most 1 committee.
How many ways to:

- Assign a leader to each committee, then distribute all $n - 5$ remaining citizens in any way?
- Assign a captain and two members to each committee?