Counting, Part I

CS 70, Summer 2019

Lecture 13, 7/16/19

Goals: Probability

- Lets you **quantify** uncertainty
- Concretely: has **applications** everywhere!
- Hopefully: learn techniques for **reasoning about randomness** and **making better decisions** logically
- Hopefully: provides a **new perspective** on the world

CS 70 Tips

The probability section in CS 70 usually means:

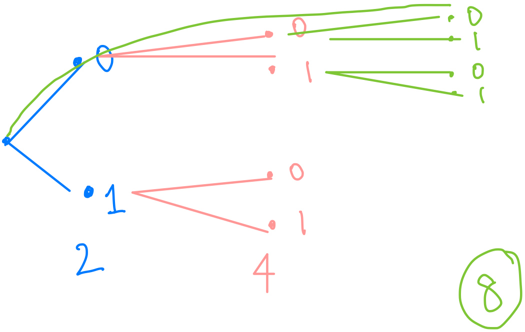
- **One big topic**, rather than many small topics
  - Try your best to stay **up to date**; use OH!
  - Important to be comfortable with the **basics**
- Fewer "proofs," more **computations**
  - Emphasis on **applying tools** and **problem solving**
  - Lectures will be **example-driven**
- **Practice, practice, practice!**

A Familiar Question

How many bit (0 or 1) strings are there of length 3?

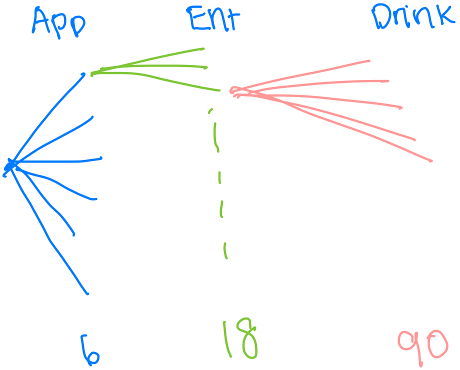
3 choices

bit $\underset{\#1}{\underline{\;2\;}} \quad \underset{\#2}{\underline{\;2\;}} \quad \underset{\#3}{\underline{\;2\;}}$



Choices, Choices, Choices...

A lunch special lets you choose one appetizer, one entreé, and one drink. There are 6 appetizers, 3 entreés, and 5 drinks. How many different meals could you possibly get?



The First Rule of Counting: Products

If the object you are counting:

- Comes from making k **choices**
- Has n_1 options for the first choice
- Has n_2 options for second, **regardless of the first**
- Has n_3 options for the third, **regardless of the first two**
- ...and so on, until the k-th choice

$\Longrightarrow$ Count the object using the **product**

$$n_1 \times n_2 \times n_3 \times \ldots \times n_k$$

Anagramming I

How many strings can we make by rearranging "CS70"?

character: $\underset{\#1}{\underline{4}} \times \underset{\#2}{\underline{3}} \times \underset{\#3}{\underline{2}} \times \underset{\#4}{\underline{1}} = 24$

$\boxed{n!} = n(n-1)(n-2)\cdots 2\cdot 1$

How many strings can we make by rearranging "ILOVECS$\boxed{70}$" if the numbers "70" must appear together in that order?

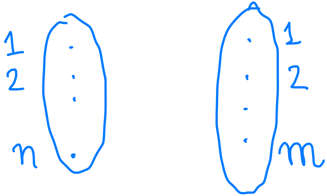
pretend it's 1 letter.

characters: $\underbrace{\underline{8} \times \underline{7} \times \underline{\ } \ \underline{\ } \ \underline{\ } \ \underline{\ } \ \underline{\ } \times \underline{1}}_{8} = 8!$

Counting Functions

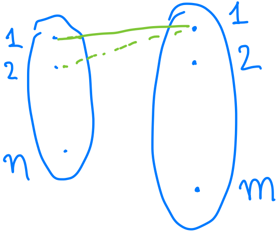
Domain — Codomain / Range

How many functions are there from $\{1, \ldots, n\}$ to $\{1, \ldots, m\}$?



$$\underset{f(1)}{m} \times \underset{f(2)}{m} \cdots \times \underset{f(n)}{m} = m^n$$

Same setup, but $m \geq n$. How many injective functions are there?



$$\underbrace{\underset{f(1)}{m} \times \underset{f(2)}{(m-1)} \times \underset{f(3)}{(m-2)} \times \cdots \underset{f(n)}{(m-n+1)}}_{n} = \frac{m!}{(m-n)!}$$

Counting Polynomials

How many degree (exactly) d polynomials are there modulo p?

$$\underbrace{\underset{x^d}{\underline{p-1}} \times \underset{x^{d-1}}{\underline{p}} \quad \cdots \quad \times \underset{x^0}{\underline{p}}}_{d+1} = (p-1)p^d$$

If $d \leq p$, how many have no repeating coefficients?

Exercise.

When Order Doesn't Matter: Space Team I

Among its 10 trainees, NASA wants to choose 3 to go to the moon. How many ways can they do this?

$$\text{people} \quad \underset{\#1}{\underline{10}} \times \underset{\#2}{\underline{9}} \times \underset{\#3}{\underline{8}} = 720 = \frac{10!}{7!}$$

ABC
ACB
BAC
BCA
CAB
CBA

6 repetitions for $\{A, B, C\}$ $\Rightarrow \frac{720}{6} = 120$

$\_\ \_\ \_ = 3!$

$$\boxed{\frac{10!}{7!\,3!}}$$

When Order Doesn't Matter: Poker I

In poker, each player is dealt 5 cards. A standard deck (no jokers) has 52 cards. How many different hands could you get?

cards $\underline{52}\ \underline{51}\ \underline{50}\ \underline{49}\ \underline{48} = \frac{52!}{47!}$

Repetitions

ABCDE
ABDEC
⋮

$5!$

$$\frac{52!}{47!\,5!}$$

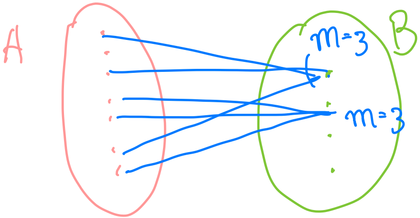
The Second Rule of Counting: Repetitions

Say we use the First Rule–we make k choices.

- Let A be the set of **ordered** objects.
- Let B be the set of **unordered** objects.

If there is an **"m-to-1"** function from A to B:

$\implies$ Count A and divide by m to get $|B|$.



Anagramming II

How many strings can we make by rearranging "APPLE"? $_{1\ 2}$

$A = \{\text{anagrams of } AP_1P_2LE\} = 5!$

m: AP_1P_2LE, AP_2P_1LE $>$ APPLE $\qquad m = 2$

$|B| \Rightarrow \dfrac{|A|}{m} = \dfrac{5!}{2} = 60.$

How many strings can we make by rearranging "BANANA"? $_{1\ 1\ 2\ 2\ 3}$

$|A| = 6!$

m: N'S: $2!$ $\qquad$ A'S: $3!$

Total: $2! \times 3!$

$|B| \Rightarrow \dfrac{|A|}{m} = \dfrac{6!}{2!\,3!}$

Binomial Coefficients

How many ways can we...

- pick a set of 2 items out of n total?

  $$\frac{n \times (n-1)}{m:\quad 2 \text{ reps.}} \quad \Bigg\} \quad \frac{n!}{(n-2)!\,2!}$$

- pick a set of 3 items out of n total?

  $$\frac{n \times (n-1) \times (n-2)}{3! \text{ reps}} \quad \Bigg\} \quad \frac{n!}{(n-3)!\,3!}$$

- pick a set of k items out of n total?

  $$\frac{n!}{(n-k)!\,k!}$$

Binomial Coefficients

We often use

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

to represent the number of ways to choose k out of n items when order doesn't matter.

We call this quantity **"n choose k"**.
We also sometimes refer to these as "binomial coefficients."

Q: Using this definition, what does 0! equal?

Binomial Coefficients

Using this definition, what does 0! equal?

$$\binom{n}{0} = 1 \qquad \frac{\cancel{n!}}{\cancel{n!}\,0!} = \frac{1}{0!} = 1$$

$$\boxed{0! = 1}$$

Should we be surprised that $\binom{n}{k} = \binom{n}{n-k}$?

$$\frac{n!}{k!\,(n-k)!} = \frac{n!}{(n-k)!\,k!}$$

ways of choosing
k members
FOR team

n-k members
NOT ON
my team!

Anagramming III

How many bit strings can we make by k 1's and $(n-k)$ 0's?

$$\underbrace{1_1 1_2 \ldots 1_k}_{k} \underbrace{0_1 0_2 \ldots 0_{n-k}}_{n-k}$$

→ with ordering: $n!$

repetitions: 1's: $k!$

0's: $(n-k)!$ $\Big\}$ $k!(n-k)!$

$$\Rightarrow \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Coincidence?

Is there a relationship between: n-k 0's

1. Length n bit strings with k 1's, and
2. Ways of choosing k items from n when order doesn't matter?

NO!!

~~Yes!~~

$$\overbrace{o \quad x \quad o \quad x \quad \cdots \quad x}^{n}$$

k x's

$n-k$ o's

$\Rightarrow$ bijection b/w $\underbrace{x/o}_{1/0}$ strings + k-item sets

Putting It All Together: Space Team II

Among its 10 trainees, NASA wants to choose 3 to go to the moon, and 2 to go to Mars. They also don't want anyone to do both missions. How many ways can they choose teams?

$$\underbrace{\binom{10}{3}}_{\text{moon}} \times \underbrace{\binom{7}{2}}_{\text{Mars}}$$

3 moon OR 2 Mars.

$$\binom{10}{3} + \binom{10}{2}$$

If one member of the moon mission is designated as a captain, how many ways can they choose teams?

$$\underbrace{\frac{10}{C}\ \frac{\binom{9}{2}}{TT}}_{\text{moon}}\ \underbrace{\binom{7}{2}}_{\text{Mars}}$$

Exercise:

$$\underbrace{\binom{10}{2}}_{\text{Mars}}\ \underbrace{\frac{\binom{8}{2}}{TT}\ \frac{6}{C}}_{\text{moon}}$$

Putting It All Together: Poker II ⊛Exercise

How many 5-card poker hands form a full house (triple + pair)?

How many 5-card poker hands form a straight (consecutive cards), including straight flushes (same suit)?

How many 5-card poker hands form two pairs?

Sampling Without Replacement

How many ways can we sample k items out of n items, **without replacement**, if:

- Order matters?

$$\underbrace{\underline{n} \times \underline{(n-1)} \times \underline{(n-2)} \times \cdots \times \underline{(n-k+1)}}_{k} = \frac{n!}{(n-k)!}$$

items

- Order does not matter?

$$\binom{n}{k}$$

(second rule of counting)
k! orderings per set.

We were able to use the First and Second rules of counting!

Sampling With Replacement

How many ways can we sample k items out of n total items, **with replacement**, if:

- Order matters?

items $\underbrace{\underline{n} \times \underline{n} \times \cdots \underline{n}}_{k} = n^k$

- Order does not matter? Label items $1, \dots, n$.

$1\ 1\ 1 \dots 1 \Rightarrow \{1, 1, \dots, 1\}$

$? \Rightarrow \{1, 1, \dots, 2\}$ ← k repetitions

What can we do when order does not matter?

When Repetitions Aren't Uniform: Splitting Money

Alice, Bob, and Charlie want to split \$6 amongst themselves.

First (naive and difficult) attempt: the "dollar's point of view"

A B C $\underline{3} \times \underline{3} \times \underline{3} \times \underline{3} \times \underline{3} \times \underline{3} = 3^6$

Dollar

* order ~~doesn't matter~~ because dollars are indistinguisnable.

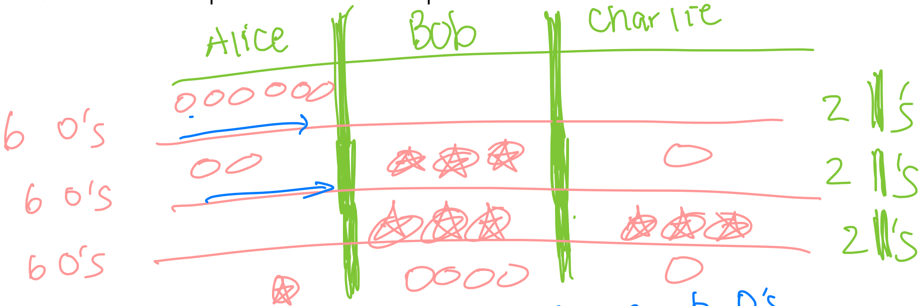
AAAAAA ✓ 1 way to get Alice 6

BAAAAA
ABAAAA
⋮
6 ways to get Alice 5 Bob 1

* CANNOT use 2nd rule!

When Repetitions Aren't Uniform: Splitting Money

Second attempt: the "divider" point of view



ways splitting \$6 $\Longleftrightarrow$ rearranging 6 0's, 2 1's

⊛ STARS AND BARS

$$\binom{8}{2} = \frac{8!}{6!\,2!}$$

"Stars and Bars" Application: Sums to k

How many ways can we choose n (not necessarily distinct) non-negative ~~numbers~~ integers that sum to k?

"with replace"

Find nonnegative x_i s.t.

$$x_1 + x_2 + \dots + x_n = k$$

$(n-1)$ |'s

k ★

$$\frac{(n-1+k)!}{(n-1)!\,k!}$$

$k \geq n$

Food for thought: What if the ~~numbers~~ integers have to be **positive**?

$\neq 0$

$x_1 \quad x_2 \quad x_3 \quad \dots \quad x_n$

0|0|00

||0000

110000

$(n-1)$ |'s

$k-n$ ★'s

$$\frac{(k-1)!}{(n-1)!\,(k-n)!}$$

Summary

- k choices, always the same number of options at choice i regardless of previous outcome $\implies$ **First Rule**
- Order doesn't matter; same number of repetitions for each desired outcome $\implies$ **Second Rule**
- Indistinguishable items split among a fixed number of different buckets $\implies$ **Stars and Bars**

Pick Your Strategy I

You have 12 distinct cards and 3 people. How many ways to:

- Deal to the 3 people in sequence (4 cards each), and the order they received the cards matters?

- Deal to the 3 people in sequence (4 cards each), but order doesn't matter?

Pick Your Strategy II

You have 12 distinct cards and 3 people. How many ways to:

- Deal 3 piles in sequence (4 cards each), and don't distinguish the piles?
- The cards are now indistinguishable. How many ways to deal so that each person receives at least 2 cards?

Pick Your Strategy III

There are n citizens on 5 different committees.
Say $n > 15$, and that each citizen is on at most 1 committee.
How many ways to:

- Assign a leader to each committee, then distribute all $n-5$ remaining citizens in any way?

- Assign a captain and two members to each committee?