Counting, Part II

CS 70, Summer 2019

Lecture 13, 7/16/19

Recap

- k choices, always the same number of options at choice i regardless of previous outcome $\implies$ **First Rule**
- Order doesn't matter; same number of repetitions for each desired outcome $\implies$ **Second Rule**
- Indistinguishable items split among a fixed number of different buckets $\implies$ **Stars and Bars**

Today: more counting strategies, and combinatorial proofs!

Count by (Disjoint) Cases: Restaurant Menu

For lunch, there are 2 appetizers, 4 entreés, and 3 desserts. The apps are salad and onion rings. If I order salad, I want both an entreé and a dessert. If I order onion rings, I only want an additional entreé. How many choices do I have for lunch?

Count by (Disjoint) Cases: Sum to 12

If $x_1, x_2, x_3 \geq 0$, how many ways can we satisfy

$$x_1 + x_2 + 5 \cdot x_3 = 12$$

Counting the Complement: Dice Rolls

If we roll 3 die, how many ways are there to get at least one 6?

First (naive, but still correct) attempt:

Counting the Complement: Dice Rolls

Second attempt:

Counting Using Symmetry: Coin Flips

How many sequences of 16 coin flips have more heads than tails?

First (naive) attempt:

Counting Using Symmetry: Coin Flips

Second attempt: Split the entire set of coin flips into three types:

1. More heads than tails
2. More tails than heads
3. Equal numbers of heads and tails

Counting Using Set Theory: Two Sets

Assume A, B, C finite sets. $|A \cup B| = |A| + |B| - |A \cap B|$
(**"A or B"** / **"at least one of A, B"**)

Applying Set Theory: Phone Numbers

How many 5-digit numbers have a 2 in the first **or** last position?

Counting Using Set Theory: Three Sets

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$
(**"A or B or C"** / **"at least one of A, B, C"**)

Complete Mixups: Warm-Up

Alice, Bob, and Charlie each bring a book to class. The books are mixed up and redistributed. How many ways could Alice, Bob, and Charlie each **not get their own book**?

How many ways can Alice **not get** her own book, with no restrictions on Bob and Charlie?

How many ways can Alice and Bob both **not get** their own book, with no restriction on Charlie?

Complete Mixups: A Realization

How many ways can Alice **get** her own book, with no restrictions on Bob and Charlie?

How many ways can Alice and Bob both **get** their own book, with no restrictions on Charlie?

The **"opposite"** problem is easier!

Complete Mixups: Finishing Argument

$A =$

$B =$

$C =$

$|A \cup B \cup C| =$

The Principle of Inclusion-Exclusion

A preview into the discrete probability section...

Say we have n subsets of a space, $A_1, \ldots, A_n$.

$$\left|\bigcup_{i=1}^{n} A_i\right| = (\text{size-1 intersections}) - (\text{size-2 intersections}) + (\text{size-3 intersections}) - \ldots$$

Intro to Combinatorial Proof: Binomial Coefficients

Powers of $(a+b)$:

$(a+b)^0 =$

$(a+b)^1 =$

$(a+b)^2 = (a+b)(a+b)$

$(a+b)^3 = (a+b)(a+b)(a+b)$

How about $(a+b)^n$? This is the **Binomial Theorem**.

Pascal's Triangle

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

- Observation #1:
- Observation #2:
- Observation #3:

Combinatorial Proof I

Observation #1: Pascal's Triangle is symmetric.
In other words: $\binom{n}{k} = \binom{n}{n-k}$

Algebraic Method:

Double-Counting Method (**"Combinatorial Proof"**):

Combinatorial Proof II

Observation #2: Adjacent elements sum to the element below.
In other words: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Algebraic Method: Try it yourself!

Double-Counting Method (**"Combinatorial Proof"**):

Combinatorial Proof III

Observation #3: Elements in row n sum to 2^n.
In other words: $\sum_{i=1}^{n} \binom{n}{i} = 2^n$

Algebraic Method: Don't try this at home!

Double-Counting Method (**"Combinatorial Proof"**):

Another Combinatorial Proof

From Notes: $\binom{n}{k+1} = \binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \ldots + \binom{k}{k}$

Algebraic Method: Don't try this at home!

Double-Counting Method (**"Combinatorial Proof"**):

Summary

- Other counting tools: casework, complements, symmetry...
- Set theory is your friend! Principle of inclusion / exclusion
- Counting problems will ask you to **decide** what tool to use and often **combine** strategies
- Combinatorial proof: count the **same thing in two ways!**