Counting, Part II

CS 70, Summer 2019

Lecture ~~13~~ 14, 7/~~16~~ 17/19

Recap

- k choices, always the same number of options at choice i regardless of previous outcome $\implies$ **First Rule**
- Order doesn't matter; same number of repetitions for each desired outcome $\implies$ **Second Rule**
- Indistinguishable items split among a fixed number of different buckets $\implies$ **Stars and Bars**

Today: more counting strategies, and combinatorial proofs!

Count by (Disjoint) Cases: Restaurant Menu

For lunch, there are 2 appetizers, 4 entreés, and 3 desserts. The apps are salad and onion rings. If I order salad, I want both an entreé and a dessert. If I order onion rings, I only want an additional entreé. How many choices do I have for lunch?

case 1: salad — Entrée 4 × Dessert 3 = 12

case 2: onion rings — Entrée 4 = 4

+ ——

16

* case 1, case 2 are disjoint

Count by (Disjoint) Cases: Sum to 12

If $x_1, x_2, x_3 \geq 0$, how many ways can we satisfy

integers

$$x_1 + x_2 + 5 \cdot x_3 = 12$$

$x_3 = \{0, 1, 2\}$

case 1 $x_3 = 0$ | case 2 $x_3 = 1$ | case 3 $x_3 = 2$

$x_1 + x_2 = 12$ | $x_1 + x_2 = 7$ | $x_1 + x_2 = 2$

13 ways to set x_1, x_2 | 8 ways | 3 ways

24 ways

Counting the Complement: Dice Rolls

If we roll 3 differently colored die, how many ways are there to get at least one 6?

First (naive, but still correct) attempt:

1 6 Dice:

$\underbrace{\underline{6}\ \underline{X}\ \underline{X}}_{\text{3 orderings.}}$

$\overset{6}{1} \times \overset{\text{non-6}}{5} \times \overset{\text{non 6}}{5} \times 3 = 75$

2 6 Dice:

$\underline{6}\ \underline{6}\ \underline{X}$

$\overset{6}{1} \times \overset{6}{1} \times \overset{\text{non 6}}{5} \times 3 = 15$

3 6 Dice:

6 6 6

1 way

91 ways

Counting the Complement: Dice Rolls

Second attempt:

complement: rolls with <u>no</u> 6

overall space: all dice rolls

$$\#\text{ at least 1 6} = |\text{overall space}| - |\text{complement}|$$

$$= \underline{6} \times \underline{6} \times \underline{6} - \underline{5} \times \underline{5} \times \underline{5}$$

$$= 216 - 125$$

$$= \underline{\underline{91 \text{ ways}}}$$

Counting Using Symmetry: Coin Flips

How many sequences of 16 coin flips have more heads than tails?

First (naive) attempt:

Cases

9H 10H 11H . . . , 16H

$\underbrace{HHH..H}_{9}\;\underbrace{TT..T}_{7}$

$\binom{16}{9} + \binom{16}{10} + \binom{16}{11} + \ldots + \binom{16}{16} = ?$

$\frac{16!}{9!\,7!}$

Counting Using Symmetry: Coin Flips

Second attempt: Split the entire set of coin flips into three types:

Goal

1. More heads than tails

Same size. Bijection

HHHT → TTTH

2. More tails than heads

3. Equal numbers of heads and tails

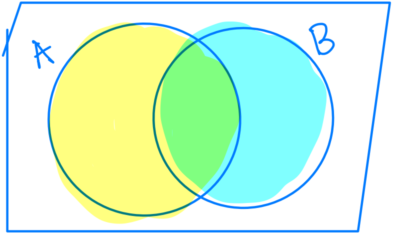
8H 8T $\binom{16}{8}$

total # seq. = 1 + 2 + 3

$2^{16} = 2x + \binom{16}{8}$

Counting Using Set Theory: Two Sets

Assume A, B, $\cancel{C}$ finite sets. $|A \cup B| = |A| + |B| - |A \cap B|$
(**"A or B"** / **"at least one of A, B"**)



$$|A \cup B| = |A| + |B| - |A \cap B|$$

Applying Set Theory: Phone Numbers

can't lead w/ 0.

How many 5-digit numbers have a 2 in the first **or** last position?

2. set union:

A : 5-dig #'s w/ 2 first

$\underline{1} \times \underline{10} \times \underline{10} \times \underline{10} \times \underline{10} =$ 10,000

B : 5-dig #'s w/ 2 last

$\underline{9} \times \underline{10} \times \underline{10} \times \underline{10} \times \underline{1}$ = 9,000

$A \cap B$: 5-dig #s w/ 2 first **and** 2 last

2

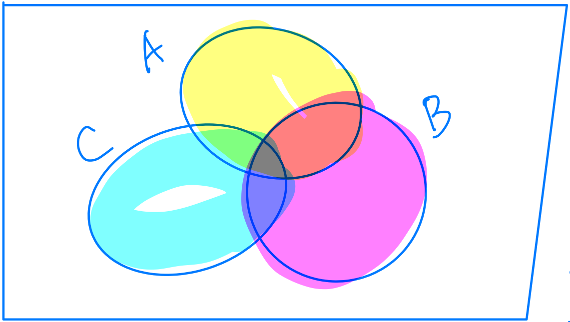
$\underline{1} \times \underline{10} \times \underline{10} \times \underline{10} \times \underline{1}$ = 1000

$|A \cup B| = 10000 + 9000 - 1000 = 18000$

Counting Using Set Theory: Three Sets

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

(**"A or B or C"** / **"at least one of A, B, C"**)



$$|A \cup B \cup C| = \underbrace{|A| + |B| + |C|}_{\text{middle counted } 3x} \underbrace{- |A \cap C| - |A \cap B| - |B \cap C|}_{\text{uncounted } 3x} + |A \cap B \cap C|$$

Complete Mixups: Warm-Up

Alice, Bob, and Charlie each bring a book to class. The books are mixed up and redistributed. How many ways could Alice, Bob, and Charlie each **not get their own book**?

How many ways can Alice **not get** her own book, with no restrictions on Bob and Charlie?

$$\underset{A}{\underline{2}} \times \underset{B}{\underline{2}} \times \underset{C}{\underline{1}} = 4 \text{ ways.}$$

How many ways can Alice and Bob both **not get** their own book, with no restriction on Charlie?

2 cases:

case 1: A gets B

$$\underset{A}{\underline{1}} \times \underset{B}{\underline{2}} \times \underset{C}{\underline{1}} = 2$$

case 2: A get C

$$\underset{A}{\underline{1}} \times \underset{B}{\underline{1}} \times \underset{C}{\underline{1}} = 1$$

Complete Mixups: A Realization

How many ways can Alice **get** her own book, with no restrictions on Bob and Charlie?

$$\underset{A}{\underline{1}} \times \underset{B}{\underline{2}} \times \underset{C}{\underline{1}} = 2 \text{ ways}$$

How many ways can Alice and Bob both **get** their own book, with no restrictions on Charlie?

$$\underset{A}{\underline{1}} \times \underset{B}{\underline{1}} \times \underset{C}{\underline{1}} = 1 \text{ ways}$$

The **"opposite"** problem is easier!

complete mixups = # rearrangements − # at least 1 gets own. ??

$$\underset{A}{\underline{3}} \times \underset{B}{\underline{2}} \times \underset{C}{\underline{1}}$$

Complete Mixups: Finishing Argument

$A =$ rearr. where Alice gets own

$B =$ " Bob "

$C =$ " Charlie "

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 2 + 2 + 2 - 1 - 1 - 1 + 1$$

prev pg.

$$= 4 \text{ ways}$$

$6 - 4 = 2$ ways

The Principle of Inclusion-Exclusion (PIE)

A preview into the discrete probability section...

Say we have n subsets of a space, $A_1, \ldots, A_n$.

$$\left| \bigcup_{i=1}^{n} A_i \right| = (\text{size-1 intersections}) - (\text{size-2 intersections})$$

$$+ (\text{size-3 intersections}) - \ldots$$

Ex: $n = 4$

$$= |A| + |B| + |C| + |D| - \left(|A \cap B| + |C \cap D| + \ldots \right)$$

$$+ \left(|A \cap D \cap C| + \ldots \right)$$

Intro to Combinatorial Proof: Binomial Coefficients

Powers of $(a+b)$:

$(a+b)^0 = 1$

$(a+b)^1 = a+b$

$(a+b)^2 = (a+b)(a+b) = a\cdot a + a\cdot b + b\cdot a + b\cdot b = a^2+2ab+b^2$

$(a+b)^3 = (a+b)(a+b)(a+b) = a\cdot a\cdot a + a\cdot a\cdot b + a\cdot b\cdot a + \ldots$

$= a^3 + 3a^2b + 3ab^2 + b^3$

aab aba baa

How about $(a+b)^n$? This is the **Binomial Theorem**.

$= a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i$

\# ways to anagram $(n-1)$ a's, 1 b

Pascal's Triangle

$(a+b)^0$ 1 $= 1$

$(a+b)^1$ 1 1 $= 2$

1 2 1 $= 4$

1 3 3 1 $= 8$

n → 1 4 6 4 1 ⋮

5 → 1 5 10 10 5 1

$\binom{5}{0}$ $\binom{5}{1}$ $\binom{5}{2}$ $\binom{5}{2}$ $\binom{5}{1}$ $\binom{5}{0}$

k, $\binom{n}{k}$

3 + 1 → 4

- Observation #1: Symmetric
- Observation #2: ○ + ○ → ○
- Observation #3: rows sum to powers of 2.

Combinatorial Proof I

Observation #1: Pascal's Triangle is symmetric.
In other words: $\binom{n}{k} = \binom{n}{n-k}$

Algebraic Method:

$$\frac{n!}{k!\,(n-k)!} = \frac{n!}{(n-k)!\,k!} \quad \checkmark$$

Double-Counting Method (**"Combinatorial Proof"**):

LHS: Choosing k members out of n for my team

RHS: Choosing $n-k$ members NOT on my team.

Combinatorial Proof II

Observation #2: Adjacent elements sum to the element below.

In other words: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ → choose out of n-1

→ choosing whole team

n-1 → o o

n → o

Algebraic Method: Try it yourself!

Double-Counting Method (**"Combinatorial Proof"**):

LHS: choose k out of n for my team.

RHS: Single out team member A

Have A

A ↓ $1 \cdot \binom{n-1}{k-1}$

people left

← A is already chosen

Don't Have A

$\binom{n-1}{k}$

Combinatorial Proof III

Observation #3: Elements in row n sum to 2^n.
In other words: $\sum_{i=\cancel{1}\,0}^{n} \binom{n}{i} = 2^n$

Algebraic Method: Don't try this at home!

Double-Counting Method (**"Combinatorial Proof"**):

LHS: $\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$

pick nobody; pick 1 pers.; pick n people

# of subsets of n people, separated by size

RHS: $\underset{\text{pers \#1}}{2} \times \underset{\text{pers \#2}}{2} \cdots \times \underset{\text{pers \# } n}{2}$ → on team or not on team.

$= 2^n$

Another Combinatorial Proof

From Notes: $\binom{n}{k+1} =$ [redacted] $\binom{n-1}{k} + \binom{n-2}{k} + \ldots + \binom{k}{k}$

choosing k: 1 fewer than $k+1$.

Algebraic Method: Don't try this at home!

Double-Counting Method (**"Combinatorial Proof"**):

LHS: choosing $(k+1)$ people out of n for my team.

RHS: Label people $1, \ldots, n$.

count by cases!

- "lowest" person #1: choose rest of team from $2, \ldots, n$
  choose k people $\binom{n-1}{k}$
- "lowest" person #2: choose rest of team from $3, \ldots, n$
  choose k people $\binom{n-2}{k}$

$\vdots$

- "lowest" person # $n-k+1$
  $\{n-k, n-k+1, \ldots, n-1, n\}$ $\qquad \binom{k}{k} = 1$

Summary

- Other counting tools: casework, complements, symmetry...
- Set theory is your friend! Principle of inclusion / exclusion
- Counting problems will ask you to **decide** what tool to use and often **combine** strategies
- Combinatorial proof: count the **same thing in two ways!**