Intro to Discrete Probability

CS 70, Summer 2019

Lecture 15, 7/18/19

Why Learn Probability?

- Uncertainty $\neq$ "nothing is known"
- Many decisions are made under **uncertainty**
  - Understanding probability gives a **precise**, **unambiguous**, **logical** way to **reason about uncertainty**
  - Also learn about **good yet simple models** for many real world situations
- Uncertainty can also be your friend!
  - We use **artificial** uncertainty to design good algorithms

Flipping Coins

I flip three coins. What is the set of outcomes?

Now, I flip n different coins. What is the set of outcomes?
How many outcomes are there?

Probability Spaces

We formalize **"experimental outcomes"** or **"samples"**:

A **probability space** is a sample space Ω, with a **probability** function $\mathbb{P}[\cdot]$ such that:

- For each sample $\omega \in \Omega$, we have

$$0 \leq \mathbb{P}(\omega) \leq 1$$

- The sum of probabilities over all $\omega \in \Omega$ is 1.

Example: Flipping Coins

I flip three **fair** coins. What is the sample space?
What are the probabilities?

Now, I flip n different **fair** coins. What is the sample space?
What are the probabilities?

Example: Flipping Coins

I flip three **biased** coins, with heads probability $p \neq \frac{1}{2}$.
What is the sample space? What are the probabilities?

Why were we able to multiply? We'll see next week...

Events

An event A is a subset of outcomes $\omega \in \Omega$.

The probability of an event A is

$$\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega]$$

Example: Flipping Coins

Let A be the event where I flip at least 2 heads.
I flip three **fair** coins. What is $\mathbb{P}[A]$?

I flip three **biased** coins, with heads probability p. What is $\mathbb{P}[A]$?

Uniform Probability Spaces

We use **"uniform"** to describe probability spaces where all outcomes have the **same probability**.

For all $\omega \in \Omega$, we have:

$$\mathbb{P}[\omega] = \frac{1}{|\Omega|}$$

As a result, for an event A:

$$\mathbb{P}[A] = \frac{|A|}{|\Omega|}$$

This is helpful for **larger** probability spaces!

Rolling Dice

I roll 2 **fair** dice.
What is the probability that both of my rolls are even?

What is the probability that both rolls are greater than 2?

Rolling Dice

What is the probability that at least one roll is less than 3?

What is the probability that the first roll is **strictly greater** than the second?

Drawing Marbles I

I have an urn with 100 marbles. Exactly 50 of them are blue and 50 of them are red.

If I draw 8 marbles from the urn **without replacement**, what is the probability I get 6 red and 2 blue?

Drawing Marbles I

If I draw 8 marbles from the urn **with replacement**, what is the probability I get 6 red and 2 blue?

Drawing Marbles II

I have an urn with 100 marbles. 50 of them are blue, 50 of them are red, and 50 of them are yellow.

If I draw 8 marbles from the urn **without replacement**, what is the probability I get 3 red, 3 blue, and 2 yellow?

Drawing Marbles II

If I draw 8 marbles from the urn **with replacement**, what is the probability I get 3 red, 3 blue, and 2 yellow?

Why didn't we use stars and bars? Discuss.

Why No Stars and Bars?

If we are running an experiment where we sample a set of objects, the outcomes counted by **stars-and-bars** is a **non-uniform** probability space.

The Birthday Problem

If there are n people in a room, what is the probability that at least two people share the same birthday?

First (naive) attempt:

The Birthday Problem

Second attempt:

The Birthday Problem: Some Stats

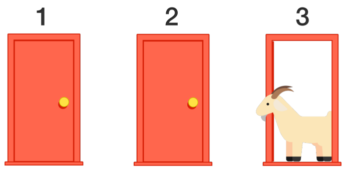
For $n = 10$, the probability is ≥ 0.11.

For $n = 23$, the probability is ≥ 0.5.

For $n = 70$, the probability is ≥ 0.999.

The Monty Hall Problem

You're on a game show. There are 3 doors you can choose from. Two of the doors lead to GOATS! One of them has a PRIZE!



You pick a door. The host then opens a **different** door that leads to a **goat**. He now gives you the option of switching to the other **unopened** door.

Poll: Should you switch?

Monty Hall: Sample Space

Each game, there are **three implicit choices** (C_1, C_2, C_3):

1. Which door leads to the prize?
2. Which door do you pick?
3. Which door does the host reveal to you?

Tree of Outcomes:

Monty Hall: Probabilities

Two cases: (1) You initially choose the prize door
(2) You initially choose a goat

(1)

(2)

Monty Hall: Events

Let W_1 = the contestant **switches doors and wins**.
Let W_2 = the contestant **stays and wins**.

Summary

- Proceed **methodically**.
  - What are the possible outcomes?
  - What is the probability for each outcome?
  - Is the sample space uniform or non-uniform?
- For **uniform** probability spaces, boils down to **counting**!