Intro to Discrete Probability

CS 70, Summer 2019

Lecture 15, 7/18/19

Why Learn Probability?

- Uncertainty $\neq$ "nothing is known"
- Many decisions are made under **uncertainty**
  - Understanding probability gives a **precise**, **unambiguous**, **logical** way to **reason about uncertainty**
  - Also learn about **good yet simple models** for many real world situations
- Uncertainty can also be your friend!
  - We use **artificial** uncertainty to design good algorithms

Flipping Coins

I flip three coins. What is the set of outcomes?

HHH $\cdot$ $\cdot$ THH

THT

8 outcomes.

Now, I flip n different coins. What is the set of outcomes?
How many outcomes are there?

⊛ strings w/ H/T, length n.

2^n outcomes.

$\cdot\,\underbrace{H \ldots H}_{n}$ $\quad \cdot\, TT\underbrace{H \ldots H}_{n-2}$

Probability Spaces

We formalize **"experimental outcomes"** or **"samples"**:

A **probability space** is a sample space Ω, with a **probability** function $\mathbb{P}[\cdot]$ such that:

Pr

- For each sample $\omega \in \Omega$, we have

$$0 \leq \mathbb{P}(\omega) \leq 1$$

- The sum of probabilities over all $\omega \in \Omega$ is 1.

Example: Flipping Coins

I flip three **fair** coins. What is the sample space?
What are the probabilities?

· 8 outcomes.

$\mathbb{P}[HHT] = \frac{1}{8}$

· HHT

· THT

$\frac{1}{8}$

Now, I flip n different **fair** coins. What is the sample space?
What are the probabilities?

2^n outcomes.

$\mathbb{P}[n \text{ H's}] = \frac{1}{2^n}$

· $\underbrace{HH\ldots H}_{n}$

Example: Flipping Coins

I flip three **biased** coins, with heads probability $p \neq \frac{1}{2}$.
What is the sample space? What are the probabilities?

same as before.

HHT THT TTT

$P(THT) = (1-p)p(1-p)$
$= p(1-p)^2$

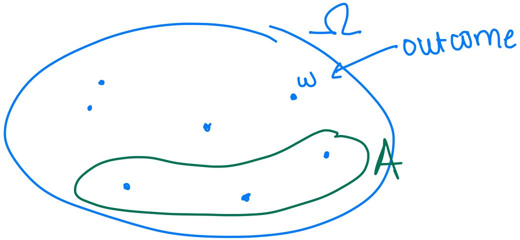
$P(TTT)$
$= (1-p)(1-p)(1-p)$
$= (1-p)^3$

Why were we able to multiply? We'll see next week...

Events

An event A is a subset of outcomes $\omega \in \Omega$.

$A \subseteq \Omega$



The probability of an event A is

$$\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega]$$

Sum of probabilities of all outcomes in A.

Example: Flipping Coins

Let A be the event where I flip at least 2 heads.
I flip three **fair** coins. What is $\mathbb{P}[A]$?

A: HHH
THH
HTH
HHT

$$\mathbb{P}[A] = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$
$$= \frac{1}{2}$$

I flip three **biased** coins, with heads probability p. What is $\mathbb{P}[A]$?

A: HHH $\quad \mathbb{P}[HHH] = p^3$
THH
HTH
HHT

$$\left.\begin{matrix} THH \\ HTH \\ HHT \end{matrix}\right\} \mathbb{P}[THH] = (1-p)p^2$$

$$\mathbb{P}[A] = p^3 + 3(1-p)p^2$$

Uniform Probability Spaces

We use **"uniform"** to describe probability spaces where all outcomes have the **same probability**.

For all $\omega \in \Omega$, we have:

$$\mathbb{P}[\omega] = \frac{1}{|\Omega|}$$

size of outcome space.

$$\mathbb{P}[A] = \sum_{\omega \in A} \underbrace{\mathbb{P}[\omega]}_{\frac{1}{|\Omega|}} = \sum_{\omega \in A} \frac{1}{|\Omega|}$$

As a result, for an event A:

$$\mathbb{P}[A] = \frac{|A|}{|\Omega|}$$

This is helpful for **larger** probability spaces!

Rolling Dice

uniform space.

Ω $\{ \cdot(1,2) \quad \cdot(5,5) \}$

$\underset{\text{Dice 1}}{6} \times \underset{\text{Dice 2}}{6} = 36 = |\Omega|$

I roll $\boxed{\text{2 **fair** dice.}}$

What is the probability that both of my rolls are even? → A

$|A| = \underset{}{\overset{2,4,6}{3}} \times \overset{2,4,6}{3} = 9$

$\mathbb{P}[A] = \frac{9}{36} = \frac{1}{4}.$

Symmetry

EE OE

EO OO

What is the probability that both rolls are greater than 2?

$|A| = 4 \times 4$

$\mathbb{P}[A] = \frac{16}{36} = \frac{4}{9}$

Rolling Dice

What is the probability that at least one roll is less than 3?

A

$$|A| = \begin{pmatrix}\#\text{ where Dice 1} \\ \text{is } < 3\end{pmatrix} + \begin{pmatrix}\#\text{ where Dice 2} \\ \text{is } < 3\end{pmatrix} - \begin{pmatrix}\#\text{ both} \\ < 3\end{pmatrix}$$

$$= 2 \times 6 + 6 \times 2 - 2 \times 2$$

$$= 12 + 12 - 4 = 20.$$

$$\mathbb{P}[A] = \frac{20}{36} = \frac{5}{9}.$$

What is the probability that the first roll is **strictly greater** than the second?

A: (*) 1) first > second
2) second > first
3) first = second.

same size.

6 ways

$$|A| + |A| + 6 = \underbrace{|\Omega|}_{36}$$

$$|A| = 15$$

$$\mathbb{P}[A] = \frac{15}{36} = \frac{5}{12}$$

Drawing Marbles I

I have an urn with 100 marbles. Exactly 50 of them are blue and 50 of them are red.

If I draw 8 marbles from the urn **without replacement**, what is the probability I get 6 red and 2 blue?

Draw sequentially

$|\Omega| = \underbrace{100 \times 99 \times 98 \cdots \times 93}_{8}$

$= \frac{100!}{92!}$

$\underbrace{RRRRRRBB}$

arrangements: $\frac{8!}{6!\,2!} = \binom{8}{6}$

$|A| = \binom{8}{6} \underbrace{50 \times 49 \times 48 \cdots \times 45}_{6} \times 50 \times 49$

$= \binom{8}{6} \cdot \frac{50!}{44!} \cdot \frac{50!}{48!}$

Drawing all at once

$|\Omega| = \binom{100}{8}$

$|A| = \underbrace{\binom{50}{6}}_{\text{red}} \underbrace{\binom{50}{2}}_{\text{blue}}$.

$\mathbb{P}[A] = \frac{\binom{50}{6}\binom{50}{2}}{\binom{100}{8}}$

Drawing Marbles I

If I draw 8 marbles from the urn **with replacement**, what is the probability I get 6 red and 2 blue?

Go sequentially:

$$|\Omega| = \underbrace{\underline{100} \times \underline{100} \times \underline{100} \ldots \times \underline{100}}_{8} = 100^8$$

sequentially => order matters: Ex: BB RRRRRR and RRRRRRBB count both

$$\Rightarrow \binom{8}{6}$$

Fix 1: $\underbrace{\underline{50} \times \underline{50} \times \underline{50} \times \underline{50} \ldots}_{6} \underbrace{\underline{50} \times \underline{50}}_{2} = 50^8$

$$|A| = \binom{8}{6} 50^8$$

$$\mathbb{P}[A] = \frac{|A|}{|\Omega|} = \frac{\binom{8}{6} 50^8}{100^8}$$

Drawing Marbles II

Exercise.

I have an urn with 100 marbles. 50 of them are blue, 50 of them are red, and 50 of them are yellow.

If I draw 8 marbles from the urn **without replacement**, what is the probability I get 3 red, 3 blue, and 2 yellow?

Drawing Marbles II

If I draw 8 marbles from the urn **with replacement**, what is the probability I get 3 red, 3 blue, and 2 yellow?

Why didn't we use stars and bars? Discuss.

Why No Stars and Bars?

If we are running an experiment where we sample a set of objects, the outcomes counted by **stars-and-bars** is a **non-uniform** probability space.

one outcome

RRRBRRRB

RRRRRRBB

$\binom{8}{6}$

BBBBBBBB

1

The Birthday Problem

If there are n people in a room, what is the probability that at least two people share the same birthday?

First (naive) attempt:

2 people same: ??

⋮

n people same: ??

Huge sum!

The Birthday Problem

Second attempt: complement: count ways no one has same birthday.

$$|\Omega| = \underbrace{\underline{365} \times \underline{365} \times \underline{365} \ldots \times \underline{365}}_{n} = 365^n.$$

complement

$$|\bar{A}| = \underbrace{\underline{365} \times \underline{364} \times \underline{363} \ldots \underline{365-n+1}}_{n} = \frac{365!}{(365-n)!}$$

$$|A| = |\Omega| - \frac{365!}{(365-n)!}$$

$$\mathbb{P}[A] = 1 - \frac{365!}{(365-n)!\,365^n}$$

The Birthday Problem: Some Stats

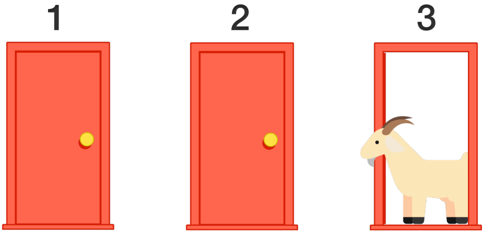
For $n = 10$, the probability is ≥ 0.11.

For $n = 23$, the probability is ≥ 0.5.

For $n = 70$, the probability is ≥ 0.999.

The Monty Hall Problem

You're on a game show. There are 3 doors you can choose from. Two of the doors lead to GOATS! One of them has a PRIZE!



You pick a door. The host then opens a **different** door that leads to a **goat**. He now gives you the option of switching to the other **unopened** door.

Poll: Should you switch?

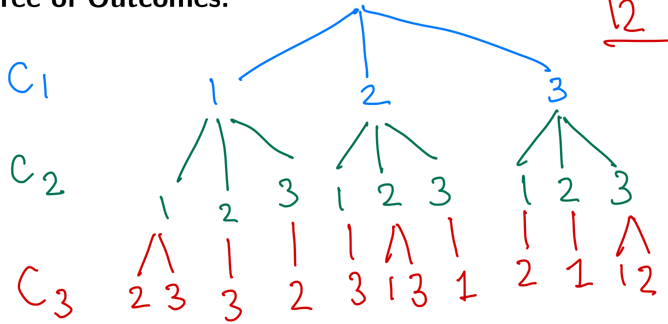
Monty Hall: Sample Space

Each game, there are **three implicit choices** (C_1, C_2, C_3):

1. Which door leads to the prize?
2. Which door do you pick?
3. Which door does the host reveal to you?

Tree of Outcomes:

12 outcomes.



Monty Hall: Probabilities

Two cases: (1) You initially choose the prize door

(2) You initially choose a goat

(1) Prob. of a yellow leaf:

$$\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{18}$$

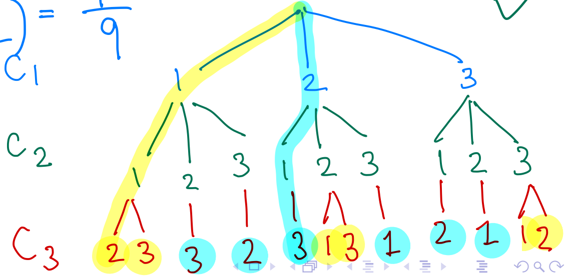
Prob of blue leaf

(2) $$\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)(1) = \frac{1}{9}$$

Sum to 1?

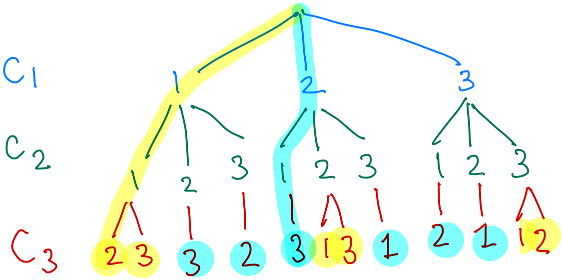
$$6\left(\frac{1}{18}\right) + 6\left(\frac{1}{9}\right)$$

$$\frac{1}{3} + \frac{2}{3} = 1$$



Monty Hall: Events

Let W_1 = the contestant **switches doors and wins**. $2/3$
Let W_2 = the contestant **stays and wins**. $1/3$



Summary

- Proceed **methodically**.
  - What are the possible outcomes?
  - What is the probability for each outcome?
  - Is the sample space uniform or non-uniform?
- For **uniform** probability spaces, boils down to **counting**!