Conditional Probability

CS 70, Summer 2019

Lecture 16, 7/22/19

Review of Last Thursday

- Proceed **methodically**.
  - What are the possible outcomes?
  - What is the probability for each outcome?
  - Is the sample space uniform or non-uniform?
- For **uniform** probability spaces, boils down to **counting**!
  - Use the same techniques: First Rule, Second Rule, complements, set theory, symmetry, etc.
  - Be consistent between your numerator and denominator

Making Use of Information I

Let's play a game. We have a full, standard deck of cards. I flip the top card and if it's red, I win. If it's black, you win.

What is your probability of winning?

Now, you swipe 6 cards from the bottom of the deck when I'm not looking. **Four are black, and two are red.**

Do you still want to play the game?

Making Use of Information II

I flip 3 fair coins. What is the probability of exactly 2 heads?
Recall: uniform probability space on **8 outcomes**.

I flip my first coin, and it is a **head**.
Now, what is the probability I get exactly 2 heads?

Conditional Probability

I want to find the probability of event A, in sample space Ω.
I have **additional information** that event B is true.

We need a new sample space, $\Omega' =$

$$\mathbb{P}[A|B] =$$

Sanity Check!

What is $\mathbb{P}[B|A]$?

List two different ways to write $A \cap B$.

If we know $\mathbb{P}[A|B]$, how do we find $\mathbb{P}[\overline{A}|B]$?

What is $\mathbb{P}[A|B]$ if A, B are **disjoint**?

Pocket Aces

(From notes.) I deal two cards. What is the probability that the second is an ace, given the first is also an ace?

$A =$

$B =$

Dice Roll

I roll a red die and a blue die. Both are fair.
If I know they sum to 6, then what is the probability that the red die is odd?

$A =$

$B =$

Probability by Disjoint Cases I

I have two coins: **one fair, one biased**.
The biased coin comes up heads with probability $\frac{3}{4}$.
I pick one coin **uniformly at random**.
What is the probability I get heads?

Probability by Disjoint Cases II

Each day, the weather in Berkeley is sunny wp 0.7, cloudy wp 0.2, and rainy wp 0.1.

On a sunny day, there is a 0.2 probability I need a jacket.
On a cloudy day, this probability is 0.5.
On a rainy day, this probability is 0.8.

What is the probability that I **don't** need a jacket?

$W_1 =$

$W_2 =$

$W_3 =$

$J =$

Probability by Disjoint Cases II

Total Probability Rule: Tree View

Total Probability Rule: Set View

Let $B_1, B_2, B_3, \ldots, B_n$ be a **partition** of the space.

$$\mathbb{P}[A] = \mathbb{P}[A \cap B_1] + \mathbb{P}[A \cap B_2] + \ldots + \mathbb{P}[A \cap B_n]$$

Total Probability Rule: Algebra View

Let $B_1, B_2, B_3, \ldots, B_n$ be a **partition** of the space.

$$\mathbb{P}[A] = \mathbb{P}[A \cap B_1] + \mathbb{P}[A \cap B_2] + \ldots + \mathbb{P}[A \cap B_n]$$

Two Cases, B and $\overline{B}$:

$$\mathbb{P}[A] =$$

Tip: Label Your Information!

(From notes.) You're slated to play a match against either opponent X or opponent Y.

The probability that you play against X is 0.6.
You beat X wp 0.7. You beat Y wp 0.3

What is the probability of winning?

Consolidate and Solve!

$\mathbb{P}[W|X] =$ $\mathbb{P}[W|\overline{X}] =$

$\mathbb{P}[X] =$ $\mathbb{P}[\overline{X}] =$

$\mathbb{P}[W] =$

Bayesian Inference

I have two coins: **one fair, one biased**.
The biased coin comes up heads with probability $\frac{3}{4}$.
I got a head. What is the probability my coin was biased?

Bayes' Rule: Two Cases

We **partition** our space into two events, $B, \overline{B}$.
Say we know $\mathbb{P}[A|B]$, $\mathbb{P}[A|\overline{B}]$, and $\mathbb{P}[B]$.

$$\mathbb{P}[B|A] =$$

Bayes' Rule: Multiple Cases

We **partition** our space into events, $B_1, B_2, \ldots B_n$.
Say we know $\mathbb{P}[A|B_i]$ for all i, and $\mathbb{P}[B_i]$ for all i.

$\mathbb{P}[B|A] =$

Tip: Label Your Information!

(From notes.) A pharmaceutical company is marketing a new test for a certain disease.

1. When applied to an affected person, the test is positive wp **0.9**. It is negative wp **0.1** (**false negative**).
2. When applied to an unaffected person, the test is negative wp **0.8**. It is positive wp **0.2** (**false positive**).

Tip: Label Your Information!

The disorder affects **5%** of the population.

What is the probability that **a person is affected if they test positive**?

Consolidate and Solve!

$\mathbb{P}[P|A] =$ $\qquad$ $\mathbb{P}[P|\overline{A}] =$

$\mathbb{P}[A] =$ $\qquad$ $\mathbb{P}[\overline{A}] =$

$\mathbb{P}[A|P] =$

Summary

- The **conditional probability** of A given B (i.e. $\mathbb{P}[A|B]$) involves **restricting the sample space** to B
  - Lets you compute $\mathbb{P}[A \cap B]$ as well:

$$\mathbb{P}[A \cap B] = \mathbb{P}[A|B]\,\mathbb{P}[B] = \mathbb{P}[B|A]\,\mathbb{P}[A]$$

- "Total probability rule" is a fancy way of saying **probability by disjoint cases**
- "Bayes' Rule" is just an application of the definition of conditional probability and total probability rule.
  - Lets you **"flip"** the conditioning:
    Given information like $\mathbb{P}[A|B_i]$, compute $\mathbb{P}[B_i|A]$.