Conditional Probability

CS 70, Summer 2019

Lecture 16, 7/22/19

Review of Last Thursday

- Proceed **methodically**.
  - What are the possible outcomes?
  - What is the probability for each outcome?
  - Is the sample space uniform or non-uniform?
- For **uniform** probability spaces, boils down to **counting**!
  - Use the same techniques: First Rule, Second Rule, complements, set theory, symmetry, etc.
  - Be consistent between your numerator and denominator

Making Use of Information I

Let's play a game. We have a full, standard deck of cards. I flip the top card and if it's red, I win. If it's black, you win.

What is your probability of winning? $\frac{1}{2}$

Now, you swipe 6 cards from the bottom of the deck when I'm not looking. **Four are black, and two are red.**

Do you still want to play the game?

46 cards
22 black.

$$\mathbb{P}[\text{win}] = \frac{22}{46} = \frac{11}{23} < \frac{1}{2}$$

Making Use of Information II

I flip 3 fair coins. What is the probability of exactly 2 heads?
Recall: uniform probability space on **8 outcomes**.

HHH TTH
HHT THT
HTH HTT
THH TTT

$$\mathbb{P}[2 \text{ heads}] = \frac{3}{8}$$

I flip my first coin, and it is a **head**.
Now, what is the probability I get exactly 2 heads?

HHH ~~TTH~~
HHT ~~THT~~
HTH HTT
~~THH~~ ~~TTT~~

$$\mathbb{P}\left[2 \text{ heads} \,\middle|\, \begin{array}{c}\text{coin \#1} \\ \text{head}\end{array}\right] = \frac{2}{4} = \frac{1}{2}$$

"given"

Multiple Tosses

n coins

1st H

$$\mathbb{P}[3\text{ H total} \mid 1^{st}\text{ H}]$$

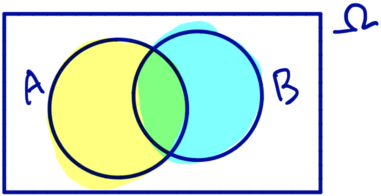
$$H\ \underbrace{\_\ \_\ \_\ \_\ \_}_{n-1,\ 2H,\ (n-3)T}$$

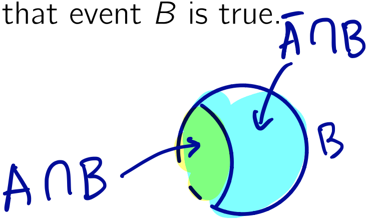
ways to get 2H, $(n-3)T$ $\iff$ Anagrams of $HHT\underbrace{T\ldots T}_{n-3}$

$$\frac{(n-1)!}{\underset{\text{H's}}{2!}\,\underset{\text{T's}}{(n-3)!}} = \binom{n-1}{2}$$

Conditional Probability

I want to find the probability of event A, in sample space Ω.
I have **additional information** that event B is true.





We need a new sample space, $\Omega' = B$

$$\boxed{\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}}$$

Sanity Check!

What is $\mathbb{P}[B|A]$?

$$\mathbb{P}[B|A] = \frac{\mathbb{P}[\underbrace{B \cap A}_{= A \cap B}]}{\mathbb{P}[A]}$$

List two different ways to write $\mathbb{P}[A \cap B]$.

$$\mathbb{P}[A \cap B] = \mathbb{P}[B|A] \cdot \mathbb{P}[A] = \mathbb{P}[A|B] \cdot \mathbb{P}[B]$$

complement

If we know $\mathbb{P}[A|B]$, how do we find $\mathbb{P}[\overline{A}|B]$?

$$\mathbb{P}[\overline{A}|B] = 1 - \mathbb{P}[A|B]$$

What is $\mathbb{P}[A|B]$ if A, B are **disjoint**?

$$\mathbb{P}[A|B] = 0$$

Pocket Aces

(From notes.) I deal two cards. What is the probability that the second is an ace, given the first is also an ace?

$A =$ 1st card A

$B =$ 2nd card A

$$\mathbb{P}[B|A] = \frac{\mathbb{P}[B \cap A]}{\mathbb{P}[A]}$$

$$\mathbb{P}[B \cap A] = \frac{4 \times 3}{52 \times 51}$$

4×3 ← # ways to get 2A

52×51 ← # ways to get 2 cards

$$\mathbb{P}[A] = \frac{4}{52} = \frac{1}{13}$$

$$\Rightarrow \mathbb{P}[B|A] = \frac{\frac{\cancel{4} \times 3}{\cancel{52} \times 51}}{\cancel{4}/\cancel{52}} = \frac{3}{51} = \frac{1}{17}$$

"Intuitive way"

After 1st A

51 left, 3 aces

$$\mathbb{P}[2^{nd} A \mid 1^{st} A] = \frac{3}{51} = \frac{1}{17}$$

Dice Roll

I roll a red die and a blue die. Both are fair.
If I know they sum to 6, then what is the probability that the red die is odd?

$A =$ dice sum to 6

$B =$ red one is odd.

Goal: compute $\mathbb{P}[B \mid A] = \dfrac{\mathbb{P}[B \cap A]}{\mathbb{P}[A]}$

$$A = \left\{ \begin{array}{cc} R & B \\ 1 & 5 \\ 2 & 4 \\ 3 & 3 \\ 4 & 2 \\ 5 & 1 \end{array} \right\}$$

$\leftarrow B \cap A.$

$= \dfrac{3}{5}$

Probability by Disjoint Cases I

I have two coins: **one fair, one biased**.

The biased coin comes up heads with probability $\frac{3}{4}$.

I pick one coin **uniformly at random**. ← each prob. $\frac{1}{2}$

What is the probability I get heads?

Let F = I chose the fair coin.

H = I flipped head.

= paths that lead to H.

$\mathbb{P}[F] = \frac{1}{2}$ → fair

$\mathbb{P}[H|F] = \frac{1}{2}$ → H

$\mathbb{P}[F \cap H] = \mathbb{P}[H|F]\mathbb{P}[F] = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$

$\mathbb{P}[\bar{H}|F] = \frac{1}{2}$ → $\bar{H}$

$\mathbb{P}[\bar{F}] = \frac{1}{2}$ → biased.

$\mathbb{P}[H|\bar{F}] = \frac{3}{4}$ → H

$\mathbb{P}[\bar{F} \cap H] = \mathbb{P}[H|\bar{F}]\mathbb{P}[\bar{F}] = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$

$\mathbb{P}[\bar{H}|\bar{F}] = \frac{1}{4}$ → $\bar{H}$

$$\boxed{\mathbb{P}[H] = \frac{1}{4} + \frac{3}{8} = \frac{5}{8}}$$

Probability by Disjoint Cases II "with probability"

Each day, the weather in Berkeley is sunny wp 0.7, cloudy wp 0.2, and rainy wp 0.1.

On a sunny day, there is a 0.2 probability I need a jacket.
On a cloudy day, this probability is 0.5.
On a rainy day, this probability is 0.8.

What is the probability that I **don't** need a jacket?

$W_1 =$ sunny $\quad \mathbb{P}[W_1] = 0.7 \quad \mathbb{P}[J \mid W_1] = 0.2$

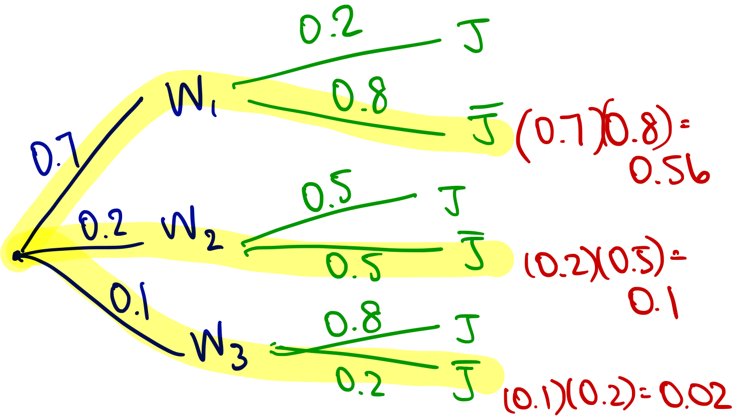
$W_2 =$ cloudy $\quad \mathbb{P}[W_2] = 0.2 \quad \mathbb{P}[J \mid W_2] = 0.5$

$W_3 =$ rainy $\quad \mathbb{P}[W_3] = 0.1 \quad \mathbb{P}[J \mid W_3] = 0.8$

$J =$ need jacket

Probability by Disjoint Cases II

$$\mathbb{P}[\bar{J}] = 0.56 + 0.1 + 0.02$$
$$= 0.68$$

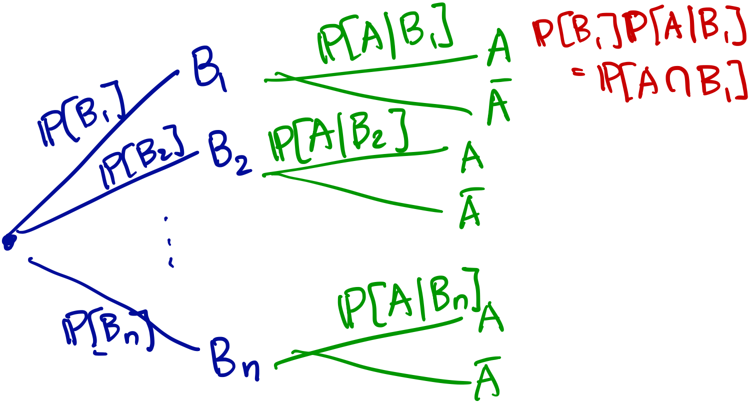


Total Probability Rule: Tree View

→ disjoint from each other
→ cover the space.

Let $\overbrace{B_1, B_2, B_3, \ldots, B_n}^{\text{cases}}$ be a **partition** of the space.

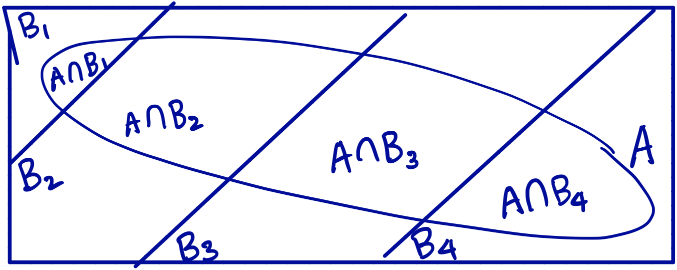
$$\mathbb{P}[A] = \mathbb{P}[A \cap B_1] + \mathbb{P}[A \cap B_2] + \ldots + \mathbb{P}[A \cap B_n]$$



Total Probability Rule: Set View

Let $B_1, B_2, B_3, \ldots, B_n$ be a **partition** of the space.

$$\mathbb{P}[A] = \mathbb{P}[A \cap B_1] + \mathbb{P}[A \cap B_2] + \ldots + \mathbb{P}[A \cap B_n]$$



Total Probability Rule: Algebra View

Let $B_1, B_2, B_3, \ldots, B_n$ be a **partition** of the space.

$$\mathbb{P}[A] = \mathbb{P}[A \cap B_1] + \mathbb{P}[A \cap B_2] + \ldots + \mathbb{P}[A \cap B_n]$$

$$= \mathbb{P}[A \mid B_1]\mathbb{P}[B_1] + \mathbb{P}[A \mid B_2]\mathbb{P}[B_2] + \ldots$$

$$= \sum_{i=1}^{n} \mathbb{P}[A \mid B_i]\mathbb{P}[B_i]$$

Two Cases, B and $\overline{B}$:

$$\mathbb{P}[A] = \mathbb{P}[A \mid B]\mathbb{P}[B] + \mathbb{P}[A \mid \overline{B}]\mathbb{P}[\overline{B}]$$

Tip: Label Your Information!

Let X = we play opponent X

W = we win!!

(From notes.) You're slated to play a match against either opponent X or opponent Y.

$\mathbb{P}[X]$

The probability that you play against X is 0.6.
You beat X wp 0.7. You beat Y wp 0.3

$\mathbb{P}[W|X]$ $\mathbb{P}[W|\bar{X}]$

What is the probability of winning?

Goal: $\mathbb{P}[W]$.

Consolidate and Solve!

$\mathbb{P}[W|X] = 0.7 \qquad \mathbb{P}[W|\overline{X}] = 0.3$

$\mathbb{P}[X] = 0.6 \qquad \mathbb{P}[\overline{X}] = 1 - \mathbb{P}[X] = 0.4$

Use Total Probability!

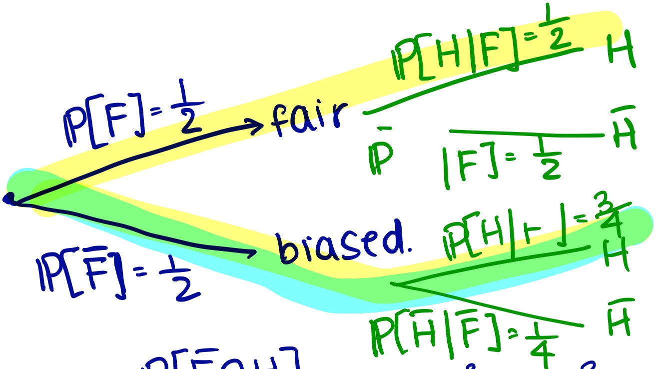
$$\mathbb{P}[W] = \mathbb{P}[W|X]\cdot\mathbb{P}[X] + \mathbb{P}[W|\bar{X}]\cdot\mathbb{P}[\bar{X}]$$

$$= (0.7)(0.6) + (0.3)(0.4)$$

$$= 0.42 + 0.12 = 0.54$$

Bayesian Inference

I have two coins: **one fair, one biased**.
The biased coin comes up heads with probability $\frac{3}{4}$.
I got a head. What is the probability my coin was biased?



$$\mathbb{P}[\bar{F} \mid H] = \frac{\mathbb{P}[\bar{F} \cap H]}{\mathbb{P}[H]} = \frac{\frac{1}{2} \cdot \frac{3}{4}}{\frac{5}{8}} = \frac{\frac{3}{8}}{\frac{5}{8}} = \frac{3}{5}$$

from prev. $\rightarrow \frac{5}{8}$

Bayes' Rule: Two Cases

We **partition** our space into two events, $B, \overline{B}$.
Say we know $\mathbb{P}[A|B]$, $\mathbb{P}[A|\overline{B}]$, and $\mathbb{P}[B]$.

$$\mathbb{P}[B|A] = \frac{\mathbb{P}[B \cap A]}{\mathbb{P}[A]}$$ ← definition.

$$= \frac{\mathbb{P}[A|B]\,\mathbb{P}[B]}{\mathbb{P}[A|B]\,\mathbb{P}[B] + \mathbb{P}[A|\overline{B}]\,\mathbb{P}[\overline{B}]}$$

← definition

↑ Total Prob. Rule.

Bayes' Rule: Multiple Cases

We **partition** our space into events, $B_1, B_2, \ldots B_n$.
Say we know $\mathbb{P}[A|B_i]$ for all i, and $\mathbb{P}[B_i]$ for all i.

$\mathbb{P}[B|A] =$

* Notes

* Same as previous, except diff form of T.P.R.

Tip: Label Your Information!

(From notes.) A pharmaceutical company is marketing a new test for a certain disease.

A = someone is affected.

P = Test is positive

1. When applied to an affected person, the test is positive wp **0.9**. It is negative wp **0.1** (**false negative**).

   $\mathbb{P}[P \mid A]$ $\mathbb{P}[\bar{P} \mid A]$

2. When applied to an unaffected person, the test is negative wp **0.8**. It is positive wp **0.2** (**false positive**).

   $\mathbb{P}[\bar{P} \mid \bar{A}]$ $\mathbb{P}[P \mid \bar{A}]$

Tip: Label Your Information!

The disorder affects **5%** of the population.

$\mathbb{P}[A] = 0.05$

"prior" – prob. w/out any observation

What is the probability that **a person is affected if they test positive**?

$\mathbb{P}[A \mid P]$

"posterior" prob. w/ observation

Consolidate and Solve!

$\mathbb{P}[P|A] = 0.9$ $\mathbb{P}[P|\overline{A}] = 0.2$

Total Prob.

$\mathbb{P}[A] = 0.05$ $\mathbb{P}[\overline{A}] = 0.95$

Apply Bayes'

$$\mathbb{P}[A|P] = \frac{\mathbb{P}[A \cap P]}{\mathbb{P}[P]} = \frac{\mathbb{P}[P|A]\,\mathbb{P}[A]}{\mathbb{P}[P|A]\,\mathbb{P}[A] + \mathbb{P}[P|\bar{A}]\,\mathbb{P}[\bar{A}]}$$

$$= \frac{(0.9)(0.05)}{(0.9)(0.05) + (0.2)(0.95)}$$

$$= \frac{0.045}{0.045 + 0.19}$$

Summary

- The **conditional probability** of A given B (i.e. $\mathbb{P}[A|B]$) involves **restricting the sample space** to B
  - Lets you compute $\mathbb{P}[A \cap B]$ as well:

$$\mathbb{P}[A \cap B] = \mathbb{P}[A|B]\,\mathbb{P}[B] = \mathbb{P}[B|A]\,\mathbb{P}[A]$$

- "Total probability rule" is a fancy way of saying **probability by disjoint cases**
- "Bayes' Rule" is just an application of the definition of conditional probability and total probability rule.
  - Lets you **"flip"** the conditioning:
    Given information like $\mathbb{P}[A|B_i]$, compute $\mathbb{P}[B_i|A]$.