Intersections and Unions of Events

CS 70, Summer 2019

Lecture 17, 7/23/19

Last Time: Conditional Probability

- $\mathbb{P}[A|B]$:**restricting the sample space** to B

$$\mathbb{P}[A \cap B] = \mathbb{P}[A|B]\,\mathbb{P}[B] = \mathbb{P}[B|A]\,\mathbb{P}[A]$$

- "Total probability rule:" **probability by disjoint cases**
- "Bayes' Rule": definition of conditional probability + total probability rule
  - Lets you **"flip"** the conditioning

Computing Intersections

For **any** events A, B:

$$\mathbb{P}[A \cap B] = \mathbb{P}[A \mid B] \cdot \mathbb{P}[B]$$
$$= \mathbb{P}[B \mid A] \cdot \mathbb{P}[A]$$

What about **any (three) events**, A, B, C?

$$\mathbb{P}[(A \cap B) \cap C] = \underbrace{\mathbb{P}[A \cap B \mid C]}\ \mathbb{P}[C]$$
$$= \mathbb{P}[A \mid B, C] \cdot \mathbb{P}[B \mid C] \cdot \mathbb{P}[C]$$
$$= \mathbb{P}[C \mid A \cap B] \cdot \underbrace{\mathbb{P}[A \cap B]}$$

Done before!

usually. $(\ast) = \mathbb{P}[C \mid A \cap B]\ \mathbb{P}[B \mid A] \cdot \mathbb{P}[A]$

Computing Intersections: Chaining

For **any** events $A_1, A_2, \ldots, A_n$:

$$\underbrace{\mathbb{P}\left[\bigcap_{i=1}^{n} A_i\right]}_{\text{intersection of } A_i} = \overbrace{\mathbb{P}[A_1]}^{A_1} \cdot \overbrace{\mathbb{P}[A_2 | A_1]}^{A_2,\text{ given } A_1} \cdot \underbrace{\mathbb{P}[A_3 | A_1 \cap A_2]}_{A_3,\text{ given } A_1, A_2} \cdot \ldots$$

$\mathbb{P}[A_4 | A_1 \cap A_2 \cap A_3]$

Proof: Details are in the notes.

General Idea: Induction!!!

Key Insight: Treat $(A_1 \cap A_2 \cap \ldots \cap A_{n-1})$ as one event, treat A_n alone as another.

$\mathbb{P}[\underbrace{(A_1 \cap \ldots \cap A_{n-1})}_{\text{one event}} \cap A_n]$ = use conditional

Very similar to...

MT1, induction question.

Drawing Cards I

I draw 4 cards sequentially from a standard deck, without replacement. What is the probability that all 4 are clubs?

$C_i =$ ith card is a club.

"$\cap$" = "and"

$\mathbb{P}[C_1 \cap C_2 \cap C_3 \cap C_4] =$

$\mathbb{P}[C_1] \cdot \mathbb{P}[C_2 | C_1] \cdot \mathbb{P}[C_3 | C_1, C_2] \cdot \mathbb{P}[C_4 | C_1, C_2, C_3]$

$= \left(\frac{13}{52}\right)\left(\frac{12}{51}\right)\left(\frac{11}{50}\right)\left(\frac{10}{49}\right)$

counting

$\mathbb{P}[4 \text{ clubs}]$

$= \frac{\#\text{ hands w/ 4 Cl.}}{\#\text{ hands}}$

$= \frac{\binom{13}{4}}{\binom{52}{4}}$

$= \frac{\frac{13!}{\cancel{4!}\,9!}}{\frac{52!}{\cancel{4!}\,48!}} = \frac{13 \cdot 12 \cdot 11 \cdot 10}{52 \cdot 51 \cdot 50 \cdot 49}$

Drawing Cards II

(Modified from notes.) I am dealt 5 cards. What is the probability that all five cards are the same suit, and none of them are face cards?

$C_1 =$ 1st card is not face.

For $2 \leq i \leq 5$: $C_i =$ ith card is not face AND same suit as first.

$\mathbb{P}[C_1 \cap C_2 \cap C_3 \cap C_4 \cap C_5] =$

$$\frac{40}{52} \cdot \frac{9}{51} \cdot \frac{8}{50} \cdot \frac{7}{49} \cdot \frac{6}{48}$$

Notes: "count" by cases

cases: by suit. $\Rightarrow 4 \cdot \mathbb{P}[5 \text{ hearts}]$

C_i = i^{th} card is a non-face spade.

$$\frac{10}{52} \cdot \frac{9}{51} \cdot \frac{8}{50} \cdot \frac{7}{49} \cdot \frac{6}{48}.$$

$\times 4$ all suits

Independent Events

For **any** events A, B:

$$\mathbb{P}[A \cap B] = \mathbb{P}[A|B] \cdot \mathbb{P}[B] = \mathbb{P}[B|A] \cdot \mathbb{P}[A]$$

doesn't have to be independent.

Two events A, B are **independent** if and only if:

$$\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$$

which is equivalent to both:

$$\mathbb{P}[A|B] = \mathbb{P}[A]$$

$$\mathbb{P}[B|A] = \mathbb{P}[B]$$

Independent or Not?

- Flipping two fair coins:
  $A =$ flip 1 is heads, $B =$ flip 2 is tails.

$\{HH, HT, TH, TT\}$

$\mathbb{P}[A] = \frac{1}{2}$ $\quad$ $\mathbb{P}[A \cap B] = \frac{1}{4}$

$\mathbb{P}[B] = \frac{1}{2}$ $\quad$ check: $\frac{1}{4} \stackrel{?}{=} \frac{1}{2} \cdot \frac{1}{2}$ ✓

- Rolling one red die, one blue die:
  $A =$ sum is 3, $B =$ red die is 1.

$\mathbb{P}[A] = \frac{2}{36} = \frac{1}{18}$ $\quad$ $\mathbb{P}[A \cap B] = \mathbb{P}[R=1, B=2] = \frac{1}{36}$

$\mathbb{P}[B] = \frac{1}{6}$

check: $\frac{1}{18} \cdot \frac{1}{6} \stackrel{?}{=} \frac{1}{36}$ ✗

Independent or Not?

- Rolling one red die, one blue die:
  A = sum is 7, B = red die is 1.

$\mathbb{P}[A] = \frac{6}{36} = \frac{1}{6}$

$\mathbb{P}[B] = \frac{1}{6}$

$\mathbb{P}[A \cap B] = \mathbb{P}[R=1, B=6] = \frac{1}{36}$

Check: $\frac{1}{36} \stackrel{?}{=} \frac{1}{6} \cdot \frac{1}{6}$ ✓

- Throwing 3 labeled balls into 3 labeled bins:
  A = Bin #1 is empty, B = Bin #2 is empty.

$\mathbb{P}[A] = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{8}{27}$

$\mathbb{P}[B] = \frac{8}{27}$

same also by symmetry.

$\mathbb{P}[A \cap B] = \mathbb{P}[\text{bin \#1, bin \#2 empty}] = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27}$

✗

Independent or Not?

A, B are generic events. The table shows probabilities of the **intersections** of the row and column.

	Event A	Event $\overline{A}$
Event B	$\mathbb{P}[A \cap B] = 0.4$	$\mathbb{P}[\overline{A} \cap B] = 0.3$
Event $\overline{B}$	$\mathbb{P}[A \cap \overline{B}] = 0.2$	$\mathbb{P}[\overline{A} \cap \overline{B}] = 0.1$

What is $\mathbb{P}[A]$? $= \mathbb{P}[A \cap B] + \mathbb{P}[A \cap \overline{B}] = 0.4 + 0.2 = 0.6$

What is $\mathbb{P}[B]$? $= \mathbb{P}[A \cap B] + \mathbb{P}[\overline{A} \cap B] = 0.4 + 0.3 = 0.7$

Are A and B independent? $0.4 \stackrel{?}{=} (0.6)(0.7)$ X

Mutual Independence

How do we generalize independence from two events A, B, to **multiple events** $A_1, A_2, \ldots, A_n$?

Definition (Mutual Independence, Ver. 1)

$A_1, A_2, \ldots, A_n$ are **mutually independent** if:

For every $I \subseteq \{1, 2, \ldots, n\}$, with $|I| \geq 2$,

any subset of $\{1,2,\ldots,n\}$

$$\mathbb{P}\left[\bigcap_{i \in I} A_i\right] = \prod_{i \in I} \mathbb{P}[A_i]$$

Ex: For 3 events:

$$\mathbb{P}\left[\bigcap_{i \in \{1,2,3\}} A_i\right] = \mathbb{P}[A_1]\,\mathbb{P}[A_2]\,\mathbb{P}[A_3].$$

Mutual Independence

Definition (Mutual Independence, Ver. 2)

$A_1, A_2, \ldots, A_n$ are **mutually independent** if:

For every choice of $B_i \in \{A_i, \overline{A_i}\}$:

$$\mathbb{P}[B_1 \cap B_2 \cap \ldots \cap B_n] = \prod_{i=1}^{n} \mathbb{P}[B_i]$$

Ex: For 3 events:

$$\mathbb{P}[A_1 \cap \bar{A}_2 \cap \bar{A}_3] = \mathbb{P}[A_1]\,\mathbb{P}[\bar{A}_2]\,\mathbb{P}[\bar{A}_3].$$

$$\mathbb{P}[\bar{A}_1 \cap A_2 \cap \bar{A}_3] = \mathbb{P}[\bar{A}_1]\,\mathbb{P}[A_2]\,\mathbb{P}[\bar{A}_3]$$

A Weaker Idea: Pairwise Independence

Definition (Pairwise Independence) $A_1, A_2, \ldots, A_n$ are

pairwise independent if:

For every $i \neq j$ in $\{1, 2, \ldots, n\}$:

$$\mathbb{P}[A_i \cap A_j] = \mathbb{P}[A_i] \cdot \mathbb{P}[A_j]$$

Q: Does mutual imply pairwise?

"every possible subset" includes pairs

Q: Does pairwise imply mutual? NO.

↳ Discussion

Using (Mutual) Independence: Fair Coin Flips

What is the probability that after n flips, we have k heads and $(n-k)$ tails?

$$\underbrace{\overbrace{H \cdots H}^{k}\ \overbrace{T \cdots T}^{n-k}}_{n}$$

$$\boxed{\mathbb{P}[k \text{ heads}, n-k \text{ T}, \text{biased}] = \binom{n}{k} p^k (1-p)^{n-k}}$$

→ # sequences k H, $(n-k)$ T.

$$= \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

coin n.

$$\mathbb{P}[\text{specific seq. } k \text{ H}, (n-k) \text{ T}] = \mathbb{P}[C_1 \text{ matches}]\ \mathbb{P}[C_2 \text{ matches}] \cdots \mathbb{P}[C_n \text{ matches}]$$

$$= \left(\frac{1}{2}\right)^n$$

What if biased? $\mathbb{P}[H] = p$

$$\mathbb{P}[\text{specific seq.}] = \underbrace{p \cdot \ldots \cdot p}_{k \text{ H}}\ \underbrace{(1-p) \cdot \ldots \cdot (1-p)}_{n-k \text{ T}}$$

$$= p^k (1-p)^{n-k}$$

Using (Mutual) Independence: Dice Rolls

We roll n red dice and n blue dice.

What is the probability that all the red dice are even, and all the blue dice are ≥ 5? Exercise.

Break

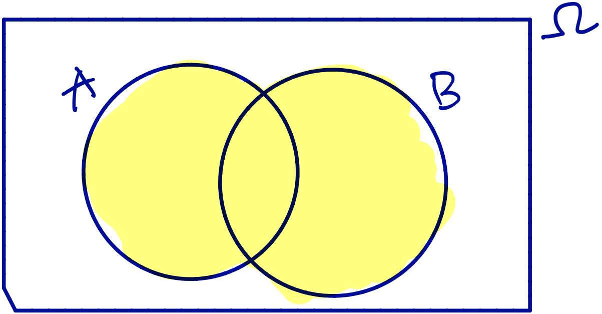
Back by popular demand...

Would you rather only use spoons (no forks) or only use forks (no spoons) for the rest of your life?

A joke... Why was 6 afraid of 7? Because seven *ate* nine. Now, why was 7 afraid of 8?

Unions of Events

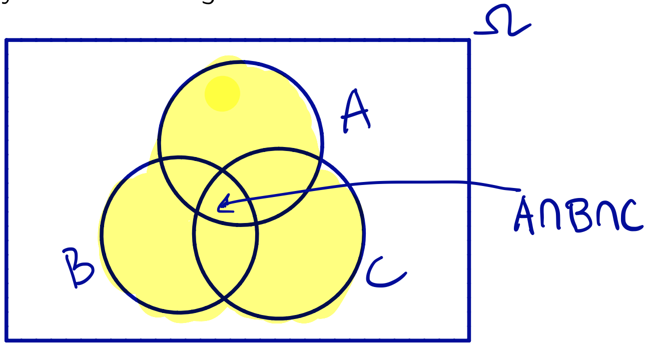
Same exact story as in counting...



$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$

Unions of Events

Same exact story as in counting...



$$\mathbb{P}[A \cup B \cup C] = \mathbb{P}[A] + \mathbb{P}[B] + \mathbb{P}[C]$$
$$- \mathbb{P}[A \cap B] - \mathbb{P}[A \cap C] - \mathbb{P}[B \cap C]$$
$$+ \mathbb{P}[A \cap B \cap C]$$

Unions Example: Rolling 3 Die

I roll a red die, a blue die, and a green die.
What is the probability that at least one of these happen?

↳ union!!

A) The red die's number is 3, or 4

B) The blue die's number is 5.

C) The green die's number is 1 or 6.

$$\mathbb{P}[A] = \frac{2}{6} = \frac{72}{6^3}$$

$$\mathbb{P}[B] = \frac{1}{6} = \frac{36}{6^3}$$

$$\mathbb{P}[C] = \frac{2}{6} = \frac{72}{6^3}$$

Example: Rolling 3 Die

Continued...

A) R 3,4,5

B) 3

C) G 1,6

$$\mathbb{P}[A \cap B] = \left(\frac{2}{6}\right)\left(\frac{1}{6}\right) = \frac{2}{36} = \frac{12}{6^3}$$

$$\mathbb{P}[A \cap C] = \left(\frac{2}{6}\right)\left(\frac{2}{6}\right) = \frac{4}{36} = \frac{24}{6^3}$$

$$\mathbb{P}[B \cap C] = \left(\frac{1}{6}\right)\left(\frac{2}{6}\right) = \frac{2}{36} = \frac{12}{6^3}$$

Subtract!

$$\mathbb{P}[A \cap B \cap C] = \left(\frac{2}{6}\right)\left(\frac{1}{6}\right)\left(\frac{2}{6}\right) = \frac{4}{6^3}$$

← add

$$\mathbb{P}[A \cup B \cup C] = \frac{72+36+72-12-24-12+4}{6^3}$$

$$= \frac{136}{216} = \frac{17}{27}$$

Principle of Inclusion and Exclusion

Same exact story as in counting...

For probability: Let $A_1, A_2, \ldots, A_n$ be events in our probability space. Denote $\{1, 2, \ldots, n\}$ by $[n]$. Then:

single events

pairwise intersections

$$\mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{\{i\}\subseteq[n]} \mathbb{P}[A_i] - \sum_{\{i,j\}\subseteq[n]} \mathbb{P}[A_i \cap A_j] + \sum_{\{i,j,k\}\subseteq[n]} \mathbb{P}[A_i \cap A_j \cap A_k] - \ldots$$

$$\ldots + (-1)^{n+1}\, \mathbb{P}[A_1 \cap A_2 \cap \ldots \cap A_n]$$

$A_1 \cup A_2 \cup \ldots \cup A_n$

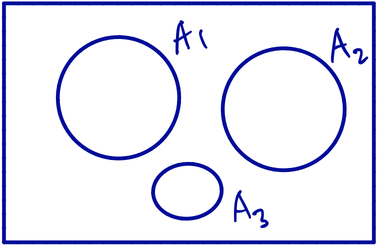
three-way intersections

The Union Bound

Q: What is the **maximum possible value** of the following?

$$\mathbb{P}[A_1 \cup A_2 \cup \ldots \cup A_n]$$

A:



$\mathbb{P}[A_1 \cup A_2 \cup A_3]$
$= \mathbb{P}[A_1] + \mathbb{P}[A_2]$
$+ \mathbb{P}[A_3]$

$\underbrace{\mathbb{P}[A_1 \cup A_2 \cup \ldots \cup A_n]}$ **is always upper bounded by**

$$\leq \mathbb{P}[A_1] + \ldots + \mathbb{P}[A_n] = \sum_{i=1}^{n} \mathbb{P}[A_i]$$

Union Bound Example: Rolling 3 Die

I roll a red die, a blue die, and a green die.
What is **an easy upper bound** on the probability that at least one of these happen?

A) The red die's number is 3 or 4.

B) The blue die's number is 5.

C) The green die's number is 1 or 6.

$$\leq \mathbb{P}[A] + \mathbb{P}[B] + \mathbb{P}[C]$$

$$= \frac{2}{6} + \frac{1}{6} + \frac{2}{6} = \frac{5}{6}$$

Actual Answer:

$$\frac{17}{27} \leq \frac{5}{6} \checkmark$$

Summary

- Computing event intersections = **chaining conditional probabilities**
  - Independent events = **directly multiply probabilities**
  - Mutual independence $\neq$ pairwise independence
- Computing event unions = same exact strategy from **counting**!
  - Draw the **Venn diagram** for 2 events, 3 events
  - Principle of Inclusion-Exclusion for multiple events
- Union bound = worst case, the events are **disjoint**!

Tips for Counting and Probability

- Don't overthink it! Consider one thing at a time.
- Label your events!! Be cognizant of **whether or not you are conditioning.**
- If you have time, **try a different strategy** and see if it gets you the same answer (e.g. cases vs. complement)
- Try **small examples** to sanity-check your strategy!
- Practice, practice, practice!