Intersections and Unions of Events

CS 70, Summer 2019

Lecture 17, 7/23/19

Last Time: Conditional Probability

- $\mathbb{P}[A|B]$:**restricting the sample space** to B

$$\mathbb{P}[A \cap B] = \mathbb{P}[A|B]\,\mathbb{P}[B] = \mathbb{P}[B|A]\,\mathbb{P}[A]$$

- "Total probability rule:" **probability by disjoint cases**
- "Bayes' Rule": definition of conditional probability + total probability rule
  - Lets you **"flip"** the conditioning

Computing Intersections

For **any** events A, B:

$$\mathbb{P}[A \cap B] =$$

What about **any (three) events**, A, B, C?

$$\mathbb{P}[A \cap B \cap C] =$$

Computing Intersections: Chaining

For **any** events $A_1, A_2, \ldots, A_n$:

$$\mathbb{P}\left[\bigcap_{i=1}^{n} A_i\right] = \mathbb{P}[A_1] \cdot \mathbb{P}[A_2|A_1] \cdot \mathbb{P}[A_3|A_1 \cap A_2] \cdot \ldots$$

Proof: Details are in the notes.

General Idea:

Key Insight: Treat $(A_1 \cap A_2 \cap \ldots \cap A_{n-1})$ as one event, treat A_n alone as another.

Very similar to...

Drawing Cards I

I draw 4 cards sequentially from a standard deck, without replacement. What is the probability that all 4 are clubs?

$C_i =$

$\mathbb{P}[C_1 \cap C_2 \cap C_3 \cap C_4] =$

Drawing Cards II

(Modified from notes.) I am dealt 5 cards. What is the probability that all five cards are the same suit, and none of them are face cards?

$C_1 =$
For $2 \leq i \leq 5$: $C_i =$

$\mathbb{P}[C_1 \cap C_2 \cap C_3 \cap C_4 \cap C_5] =$

Independent Events

For **any** events A, B:

$$\mathbb{P}[A \cap B] = \mathbb{P}[A|B] \cdot \mathbb{P}[B] = \mathbb{P}[B|A] \cdot \mathbb{P}[A]$$

Two events A, B are **independent** if and only if:

$\mathbb{P}[A \cap B] =$

which is equivalent to both:

$\mathbb{P}[A|B] =$

$\mathbb{P}[B|A] =$

Independent or Not?

- Flipping two fair coins:
  A = flip 1 is heads, B = flip 2 is tails.
- Rolling one red die, one blue die:
  A = sum is 3, B = red die is 1.

Independent or Not?

- Rolling one red die, one blue die:
  A = sum is 7, B = red die is 1.
- Throwing 3 labeled balls into 3 labeled bins:
  A = Bin #1 is empty, B = Bin #2 is empty.

Independent or Not?

A, B are generic events. The table shows probabilities of the **intersections** of the row and column.

	Event A	Event $\overline{A}$
Event B	$\mathbb{P}[A \cap B] = 0.4$	$\mathbb{P}[\overline{A} \cap B] = 0.3$
Event $\overline{B}$	$\mathbb{P}[A \cap \overline{B}] = 0.2$	$\mathbb{P}[\overline{A} \cap \overline{B}] = 0.1$

What is $\mathbb{P}[A]$?

What is $\mathbb{P}[B]$?

Are A and B independent?

Mutual Independence

How do we generalize independence from two events A, B, to **multiple events** $A_1, A_2, \ldots, A_n$?

Definition (Mutual Independence, Ver. 1)

$A_1, A_2, \ldots, A_n$ are **mutually independent** if:

For every $I \subseteq \{1, 2, \ldots, n\}$, with $|I| \geq 2$,

$$\mathbb{P}\left[\bigcap_{i \in I} A_i\right] = \prod_{i \in I} \mathbb{P}[A_i]$$

Ex: For 3 events:

Mutual Independence

Definition (Mutual Independence, Ver. 2)

$A_1, A_2, \ldots, A_n$ are **mutually independent** if:

For every choice of $B_i \in \{A_i, \overline{A_i}\}$:

$$\mathbb{P}[B_1 \cap B_2 \cap \ldots \cap B_n] = \prod_{i=1}^{n} \mathbb{P}[B_i]$$

Ex: For 3 events:

A Weaker Idea: Pairwise Independence

Definition (Pairwise Independence) $A_1, A_2, \ldots, A_n$ are

pairwise independent if:

For every $i \neq j$ in $\{1, 2, \ldots, n\}$:

$$\mathbb{P}[A_i \cap A_j] = \mathbb{P}[A_i] \cdot \mathbb{P}[A_j]$$

Q: Does mutual imply pairwise?

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Using (Mutual) Independence: Coin Flips

What is the probability that after n flips, we have k heads and $(n-k)$ tails?

Using (Mutual) Independence: Dice Rolls

We roll n red dice and n blue dice.
What is the probability that all the red dice are even, and all the blue dice are ≥ 5?

Break

Back by popular demand...

Would you rather only use spoons (no forks) or only use forks (no spoons) for the rest of your life?

A joke... Why was 6 afraid of 7? Because seven *ate* nine.
Now, why was 7 afraid of 8?

Unions of Events

Same exact story as in counting...

$\mathbb{P}[A \cup B] =$

Unions of Events

Same exact story as in counting...

$\mathbb{P}[A \cup B \cup C] =$

Unions Example: Rolling 3 Die

I roll a red die, a blue die, and a green die.
What is the probability that at least one of these happen?

A) The red die's number is 3, or 4

B) The blue die's number is 5.

C) The green die's number is 1 or 6.

$\mathbb{P}[A] =$

$\mathbb{P}[B] =$

$\mathbb{P}[C] =$

Example: Rolling 3 Die

Continued...

$\mathbb{P}[A \cap B] =$

$\mathbb{P}[A \cap C] =$

$\mathbb{P}[B \cap C] =$

$\mathbb{P}[A \cap B \cap C] =$

$\mathbb{P}[A \cup B \cup C] =$

Principle of Inclusion and Exclusion

Same exact story as in counting...

For probability: Let $A_1, A_2, \ldots, A_n$ be events in our probability space. Denote $\{1, 2, \ldots, n\}$ by $[n]$. Then:

$$\mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{\{i\}\subseteq[n]} \mathbb{P}[A_i] - \sum_{\{i,j\}\subseteq[n]} \mathbb{P}[A_i \cap A_j] + \sum_{\{i,j,k\}\subseteq[n]} \mathbb{P}[A_i \cap A_j \cap A_k] - \ldots \ldots + (-1)^{n+1}\,\mathbb{P}[A_1 \cap A_2 \cap \ldots \cap A_n]$$

The Union Bound

Q: What is the **maximum possible value** of the following?

$$\mathbb{P}[A_1 \cup A_2 \cup \ldots \cup A_n]$$

A:

$\mathbb{P}[A_1 \cup A_2 \cup \ldots \cup A_n]$ **is always upper bounded by**

Union Bound Example: Rolling 3 Die

I roll a red die, a blue die, and a green die.
What is **an easy upper bound** on the probability that at least one of these happen?

A) The red die's number is 3 or 4.

B) The blue die's number is 5.

C) The green die's number is 1 or 6.

Summary

- Computing event intersections = **chaining conditional probabilities**
  - Independent events = **directly multiply probabilities**
  - Mutual independence ≠ pairwise independence
- Computing event unions = same exact strategy from **counting**!
  - Draw the **Venn diagram** for 2 events, 3 events
  - Principle of Inclusion-Exclusion for multiple events
- Union bound = worst case, the events are **disjoint**!

Tips for Counting and Probability

- Don't overthink it! Consider one thing at a time.
- Label your events!! Be cognizant of **whether or not you are conditioning.**
- If you have time, **try a different strategy** and see if it gets you the same answer (e.g. cases vs. complement)
- Try **small examples** to sanity-check your strategy!
- Practice, practice, practice!