### Intro to Random Variables

CS 70. Summer 2019

Lecture 18, 7/24/19

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# **Definition: Random Variable**

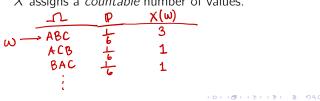
 $\rho$  ρτοβωδίμτίεο Let  $\Omega$ ,  $\mathbb P$  correspond to a probability space.

A random variable X is a function!

For every outcome, X assigns it a **real number**.

### Discrete random variable:

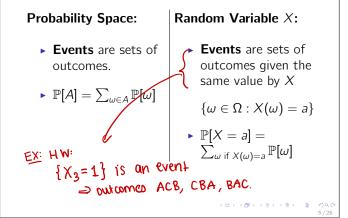
X assigns a *countable* number of values.



### **Questions**

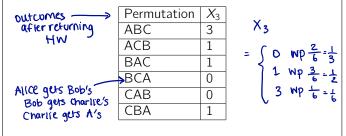
- ▶ If I flip 20 coins, how many are heads?
- ▶ If I enter a raffle with 9 other people every day, when will I first win?
- ▶ If I pick a random woman from the US population, what is her height?
- ▶ If I mix up Alice, Bob, and Charlie's HW before returning them, how many of them will get their own HW back?

# **Connections to Probability Intro**



# **Example: Returning HW**

(From notes.) Let  $X_3$  = the number of **fixed points** 



### **Definition: Distribution**

The **distribution** of a random variable X consists of two things:

- ightharpoonup The **values** X can take on. HW example: X3 = # of fixed points, 3 students {0,1,3}
- ▶ The **probability** of each value. HW example:

$$\begin{cases}
P[X_3 = 0] = \frac{2}{6} = \frac{1}{3} \\
P[X_3 = 1] = \frac{1}{2} \\
P[X_3 = 3] = \frac{1}{6}
\end{cases}$$

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## **Sanity Check!**

- ▶ What should the probabilities sum to, across all values X can take on? 1
- $\triangleright$  Can X take on **negative** values?  $\Upsilon$  est
- ▶ Can *X* take on an **infinite number** of values?
  - Raffle: can win for the Countable values? first time after any # of days

Uncountable values?

-Height in a population continuous RVs -Darts! (Later.)

### **Functions of RVs**

Same definition for X

$$X = \begin{cases} 1 & \text{wp 0.4} \\ \frac{1}{2} & \text{wp 0.25} \\ -\frac{1}{2} & \text{wp 0.35} \end{cases}$$

Write the distribution of  $X^2$ .

$$\chi^{2} = \begin{cases} 1 & \text{wp. 0.4} \\ \frac{1}{4} & \text{wp. 0.25} \end{cases} = \begin{cases} 1 & \text{wp. 0.4} \\ \frac{1}{4} & \text{wp. 0.5} \end{cases}$$

## Working with RVs

Let X be a random variable with the following distribution:

$$X = \begin{cases} 1 & \text{wp 0.4} \\ \frac{1}{2} & \text{wp 0.25} \\ -\frac{1}{2} & \text{wp 0.35} \end{cases}$$

What is the probability that *X* is **positive**?

$$P[x>0] = P[x=1] + P[x - \frac{1}{2}]$$
  
= 04+0.25 = 0.65



### Bernoulli Random Variable

Models whether **one biased coin flip** is a head. ⇒ Models a **yes/no**-type question or event

Possible values of 
$$X$$
:  $\{0,1\}$  " $ye$ "
$$\mathbb{P}[X=1] = p$$

$$\mathbb{P}[X=0] = [-p]$$

Parameters: p

Notation:  $X \sim \text{Bernoulli}(p)$ 

### **Functions of RVs**

Same definition for X

$$X = \begin{cases} 1 & \text{wp } 0.4 \\ \frac{1}{2} & \text{wp } 0.25 \\ -\frac{1}{2} & \text{wp } 0.35 \end{cases}$$

Write the distribution of f(X), where  $f: \mathbb{R} \to \mathbb{R}$ .

$$f(\chi) = \begin{cases} f(1) & \text{wp 0.4} \\ f(\frac{1}{2}) & \text{wp 0.25} \\ f(-\frac{1}{2}) & \text{wp 0.35} \end{cases}$$
also a RV!!

# **Bernoulli Example: Indicators**

If  $X \sim \text{Bernoulli}(p)$ , and X = 1 corresponds to an event A in an experiment:

 $\implies$  We say that X is an **indicator** for A.

Each day, if it is sunny in Berkeley with probability 0.8 and cloudy with probability 0.2. Indicator for a sunny day?

$$S =$$
 indicator for a sunny day  $S = \begin{cases} 1 & \text{wp. 0.8} \\ 0 & \text{wp. 0.2} \end{cases}$ 

### **Binomial Random Variable**

Models how many heads are in *n* biased coin flips

P-TH79 ⇒ Models a sum of independent, identically distributed (i.i.d) Bernoulli(p) RVs.

Possible values of  $X: \{0, 1, ..., n\}$ 

$$\mathbb{P}[X=i] = \mathbb{I} P\left[ \underset{i \in \mathcal{I}}{\text{first in flips}} \right] = \binom{n}{i} p^{i} \left( 1 - p^{i} \right)$$

**Parameters**: n, p

Notation:  $X \sim \text{Bin}(n, p)$ 

### **Break**

What is your real favorite movie, and what movie do you pretend is your favorite to sound cultured?

# Binomial Example: Weather I

Each day, it is sunny in Berkeley with probability 0.8 and cloudy with probability 0.2. Weather across days is independent.

What is the probability that over a 10 day period, there are exactly 5 sunny days?

$$S = \#$$
 sunny over to days.  
 $S \sim Bin(n,p)$   $n = 10$ ,  $p = 0.8$ 

$$P[S=5] = {10 \choose 5} (0.8)^{5} (0.2)^{5}$$

$$(n) P^{i} (1-P)^{n-i}$$

$$Ber(0.8). S_{i} = 1 \text{ # day i is } S=S_{1}+S_{2}+...+S_{n}$$

### **Geometric Random Variable**

Models how many biased coin flips I need until IP[H]=p. my first head.

⇒ Models time until a "success" when performing **i.i.d.** trials with **success** probability p

Possible values of  $X: 1, 2, 3, \dots$ 

e values of 
$$X: 1, 2, 3, \dots$$
 integers.
$$\mathbb{P}[X = i] = \mathbb{P}\begin{bmatrix} i-1 & \text{fail}, \\ \text{tren one} \\ \text{success} \end{bmatrix} = (I-P)(I-P)...(I-P)p$$

**Parameters**: **Success** probability *p* 

Notation:  $X \sim \text{Geometric}(p)$ 

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# Binomial Example: Weather II

What is the probability that over a 10 day period, there are at least two sunny days?

$$|P[S \ge 2] = 1 - |P[S \le 1] \leftarrow complement$$

$$= 1 - |P[S = 0] - |P[S = 1]$$

$$use distribution Bin(10, 0.8)$$

$$|P[S=0] = \binom{10}{0} (0.8)^{0} (0.2)^{10} = (0.2)^{10}$$

$$|P[S=1] = \binom{10}{0} (0.8)^{2} (0.2)^{3} = 10 \cdot 0.8 \cdot 0.2^{3}$$

### **Probabilities Sum To 1?** X~ Geom(P)

Not obvious that the probabilities sum to 1.

$$\sum_{i=1}^{\infty} (1-p)^{i-1}p = p + (1-p) p + (1-p)^{2}p + \dots$$

Each term is the previous  $\times$  the **same** multiplier.  $\{a, ar, ar^2, ar^3, \ldots\}$  is a **geometric sequence**.

Starting "ratio" 
$$\sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1-r} = \frac{P}{1-(1-P)}$$
Need  $|\Gamma| < 1$   $|$ 

### Aside: The Formula

An easy way to recreate the formula?

Let 
$$S = a + ar + ar^2 + \dots$$
  $\leftarrow$  Sum of all terms.

Key Idea: distributing r

$$r \cdot S = ar + ar^2 + ar^3 + \dots = S - Q$$

Solve for S! 
$$rS=S-\alpha$$

$$\alpha = S-Sr$$

$$\alpha = S(1-r)$$

$$S = \frac{\alpha}{1-r}$$

If  $|r| \ge 1$ , " $S = \infty$ "  $\rightarrow doesn't work.$ 

### **Poisson Random Variable**

Models number of rare events over a time period ⇒ Use the "rate" of event per unit time.

Possible values of 
$$X: 0, 1, 2, ...$$
 non-neg. integers.
$$\mathbb{P}[X = i] = \frac{\lambda^i}{i!} e^{-\lambda}$$

Parameters: Rate  $\lambda$ 

Notation:  $X \sim Poisson(\lambda)$ 

# Geometric Example: Raffle I

I enter a raffle with 9 other people every day. Each day, a winner is chosen independently, and with equal probability.

What is the probability that I win **for the first time** on the 5th day?

$$W = day$$
 of first win.  
 $p = \frac{1}{10}$ 

### **Probabilities Sum To 1?**

Again, not obvious that the probabilities sum to 1.

$$\sum_{i=0}^{\infty} \frac{\lambda^{i}}{i!} e^{-\lambda} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^{i}}{i!}$$

$$= e^{\lambda} \cdot e^{-\lambda} = 1$$

Taylor Series for  $e^x$ :

$$e^{x} = \sum_{i=0}^{\infty} \frac{x^{i}}{i!}$$

# Geometric Example: Raffle II for the same

What is the probability that I win the raffle some time on or before the 8th day??

$$P[W \le 8] = 1 - P[W \ge 9]$$

$$= 1 - P[\log e^{x} \cos e^{x}]$$

$$= 1 - (\frac{9}{10})^{8}$$

If  $X \sim \text{Geometric}(p)$ , then:

$$\mathbb{P}[X \ge i] = \mathbb{P}\left[\begin{array}{c} i-1 \\ i \text{ osses} \end{array}\right] = \left(1-p\right)^{i-1}$$

# Poisson Example: Typos

(Notes.) We make on average 1 typo per page. The number of typos per page is modeled by a Poisson( $\lambda$ ) random variable.

What is the probability that a single page has exactly 5 typos?

$$T = \# \text{ typos per page.}$$
 $\lambda = 1$ 

$$P[T=5] = \frac{\lambda^5}{5!} e^{-\lambda} = \frac{1}{120} e^{-1} = \frac{1}{1200}$$

# **Poisson Example: Typos**

We type 200 pages. **The pages are all independent.** What is the probablity that at least one page has **exactly** 5 typos?

one page has **exactly** 5 typos?

$$P[at | east 1 Pg.] = 1 - P[no pages]$$

$$= 1 - [P[1 page]]^{200}$$

$$= 1 - [] - \frac{1}{1200}^{200}$$

$$= revious$$

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### **Summary**

- Random variables assign numbers to outcomes.
- ▶ Treat X = i as any ordinary event.
- ► Bernoulli, Binomial, and Geometric RVs have nice interpretations via **biased coin flips**.
- ► Practice **modeling** real world events as Bernoulli, Binomial, Geometric, Poisson.

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