Intro to Random Variables

CS 70, Summer 2019

Lecture 18, 7/24/19

Questions

- If I flip 20 coins, **how many are heads**?
- If I enter a raffle with 9 other people every day, **when will I first win**?
- If I pick a random woman from the US population, **what is her height**?
- If I mix up Alice, Bob, and Charlie's HW before returning them, **how many of them will get their own HW back**?

Example: Returning HW

(From notes.)
Let X_3 = the number of **fixed points**

Permutation	X_3
ABC	3
ACB	1
BAC	1
BCA	0
CAB	0
CBA	1

Definition: Random Variable

Let $\Omega, \mathbb{P}$ correspond to a probability space.

A **random variable** X is a **function**!

For every outcome, X assigns it a **real number.**

Discrete random variable:
X assigns a *countable* number of values.

Connections to Probability Intro

Probability Space:

- **Events** are sets of outcomes.
- $\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega]$

Random Variable X:

- **Events** are sets of outcomes given the same value by X

  $\{\omega \in \Omega : X(\omega) = a\}$
- $\mathbb{P}[X = a] = \sum_{\omega \text{ if } X(\omega)=a} \mathbb{P}[\omega]$

Definition: Distribution

The **distribution** of a random variable X consists of two things:

- The **values** X can take on.
  HW example:
- The **probability** of each value.
  HW example:

Sanity Check!

- What should the probabilities sum to, across **all values** X can take on?
- Can X take on **negative** values?
- Can X take on an **infinite number** of values?
  - *Countable* values?
  - *Uncountable* values?

Working with RVs

Let X be a random variable with the following distribution:

$$X = \begin{cases} 1 & \text{wp } 0.4 \\ \frac{1}{2} & \text{wp } 0.25 \\ -\frac{1}{2} & \text{wp } 0.35 \end{cases}$$

What is the probability that X is **positive**?

Functions of RVs

Same definition for X

$$X = \begin{cases} 1 & \text{wp } 0.4 \\ \frac{1}{2} & \text{wp } 0.25 \\ -\frac{1}{2} & \text{wp } 0.35 \end{cases}$$

Write the distribution of $f(X)$, where $f : \mathbb{R} \to \mathbb{R}$.

Functions of RVs

Same definition for X

$$X = \begin{cases} 1 & \text{wp } 0.4 \\ \frac{1}{2} & \text{wp } 0.25 \\ -\frac{1}{2} & \text{wp } 0.35 \end{cases}$$

Write the distribution of X^2.

Bernoulli Random Variable

Models whether **one biased coin flip** is a head.
$\Longrightarrow$ Models a **yes/no**-type question or event

Possible values of X:

$$\mathbb{P}[X = \quad] = p$$

Parameters: p

Notation: $X \sim \text{Bernoulli}(p)$

Bernoulli Example: Indicators

If $X \sim \text{Bernoulli}(p)$, and $X = 1$ corresponds to an event A in an experiment:
$\Longrightarrow$ We say that X is an **indicator** for A.

Each day, if it is sunny in Berkeley with probability 0.8 and cloudy with probability 0.2.
Indicator for a sunny day?

Binomial Random Variable

Models how many heads are in n **biased coin flips**

$\implies$ Models a sum of independent, identically distributed (**i.i.d**) Bernoulli(p) RVs.

Possible values of X:

$$\mathbb{P}[X = i] =$$

Parameters: n, p

Notation: $X \sim \text{Bin}(n, p)$

Binomial Example: Weather I

Each day, if it is sunny in Berkeley with probability 0.8 and cloudy with probability 0.2.
Weather across days is independent.

What is the probability that over a 10 day period, there are exactly 5 sunny days?

$$n = \qquad , p =$$

Binomial Example: Weather II

What is the probability that over a 10 day period, there are at least two sunny days?

Break

What is your real favorite movie, and what movie do you pretend is your favorite to sound cultured?

Geometric Random Variable

Models how many **biased coin flips** I need until my first head.

$\implies$ Models time until a "success" when performing **i.i.d.** trials with **success** probability p

Possible values of X:

$$\mathbb{P}[X = i] =$$

Parameters: **Success** probability p

Notation: $X \sim \text{Geometric}(p)$

Probabilities Sum To 1?

Not obvious that the probabilities sum to 1.

$$\sum_{i=1}^{\infty} (1-p)^{i-1} p =$$

Each term is the previous $\times$ the **same** multiplier.
$\{a, ar, ar^2, ar^3, \ldots\}$ is a **geometric sequence**.

$$\sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1-r}$$

Aside: The Formula

An easy way to recreate the formula?

Let $S = a + ar + ar^2 + \dots$

Key Idea:

$$r \cdot S = ar + ar^2 + ar^3 + \cdots =$$

Solve for S!

Geometric Example: Raffle I

I enter a raffle with 9 other people every day. Each day, a winner is chosen independently, and with equal probability.

What is the probability that I win **for the first time** on the 5th day?

$$p =$$

Geometric Example: Raffle II

What is the probability that I win the raffle some time **on or before** the 8th day??

If $X \sim \text{Geometric}(p)$, then:

$$\mathbb{P}[X \quad i] =$$

Poisson Random Variable

Models number of **rare events** over a time period $\implies$ Use the "rate" of event per unit time.

Possible values of X:

$$\mathbb{P}[X = i] = \frac{\lambda^i}{i!} e^{-\lambda}$$

Parameters: **Rate** λ

Notation: $X \sim \text{Poisson}(\lambda)$

Probabilities Sum To 1?

Again, not obvious that the probabilities sum to 1.

$$\sum_{i=1}^{\infty} \frac{\lambda^i}{i!} e^{-\lambda} =$$

Taylor Series for e^x:

$$e^x = \sum_{i=1}^{\infty} \frac{x^i}{i!}$$

Poisson Example: Typos

(Notes.) We make on average 1 typo per page. The number of typos per page is modeled by a $\text{Poisson}(\lambda)$ random variable.

What is the probability that a single page has exactly 5 typos?

$$\lambda =$$

Poisson Example: Typos

We type 200 pages. **The pages are all independent.** What is the probablity that at least one page has **exactly** 5 typos?

Summary

- Random variables **assign numbers** to outcomes.
- Treat $X = i$ as any ordinary event.
- Bernoulli, Binomial, and Geometric RVs have nice interpretations via **biased coin flips**.
- Practice **modeling** real world events as Bernoulli, Binomial, Geometric, Poisson.