RVs Continued: Joint Distribution and Intro to Expectation

CS 70, Summer 2019

Lecture 19, 7/25/19

From Yesterday...

- RVs **assign numbers** to outcomes.
- **Treat $X = i$ as any ordinary event.**
- Bernoulli, Binomial, Geometric, Poisson RVs.

Today:

- Joint Distributions, Independent RVs, Conditional Probability
- Introduction to expectation and linearity of expectation

Joint Distribution

Lets you work with **multiple** random variables.
No different from **intersections of events!**

RV X: takes **values** a in **set** A
RV Y: takes **values** b in **set** B

Joint Distribution:
Values:

Specify the Probabilities:

Joint Distribution: Example I

	$X = 1$	$X = 2$	$X = 3$
$Y = 2$	0.2	0.2	0.1
$Y = 3$	0.1	0	0.3
$Y = 4$	0	0.1	0

$\mathbb{P}[(X \text{ even}) \cap (Y \text{ even})] =$

Joint Distribution: Example II

	$X = 1$	$X = 2$	$X = 3$
$Y = 2$	0.2	0.2	0.1
$Y = 3$	0.1	0	0.3
$Y = 4$	0	0.1	0

$\mathbb{P}[X = 2] =$

$\mathbb{P}[Y = 2] =$

$\mathbb{P}[(X = 2) \cap (Y = 2)] =$

Joint Distribution: Example III

	$X = 1$	$X = 2$	$X = 3$
$Y = 2$	0.2	0.2	0.1
$Y = 3$	0.1	0	0.3
$Y = 4$	0	0.1	0

Are the events $X = 1$ and $Y = 2$ **independent**?

Independent Random Variables

RVs X (values in A) and Y (values in B) are **independent** if:

for all $a \in A, b \in B$:

$$\mathbb{P}[X = a, Y = b] =$$

Essentially the same story as **ordinary events**!!

Conditional Distributions

Also the **same exact story**:

	$X = 1$	$X = 2$	$X = 3$
$Y = 2$	0.2	0.2	0.1
$Y = 3$	0.1	0	0.3
$Y = 4$	0	0.1	0

$\mathbb{P}[(Y \text{ even}) | X \leq 2] =$

Memorylessness of Geometrics

Memoryless: For all positive integers s, t:

$$\mathbb{P}[X \geq s + t | X > t] = \mathbb{P}[X \geq s]$$

Let $X \sim \text{Geometric}(p)$. X is memoryless:

Sum of Two Independent Poissons

Let $X \sim \text{Poisson}(\lambda_1)$, $Y \sim \text{Poisson}(\lambda_2)$.
X and Y are independent.

$\mathbb{P}[X + Y = k] =$

Break

Which building on or near campus is your "spirit building"?

Expectation of a RV

Also called the **mean** or **average** of a RV.
Let X be a RV with values in A.

Its **expectation** is defined as:

$$\mathbb{E}[X] =$$

Expectation of a RV: Example I

$$X = \begin{cases} 1 & \text{wp } 0.4 \\ \frac{1}{2} & \text{wp } 0.25 \\ -\frac{1}{2} & \text{wp } 0.35 \end{cases}$$

$\mathbb{E}[X] =$

Expectation of a RV: Example III

	$X = 1$	$X = 2$	$X = 3$
$Y = 2$	0.2	0.2	0.1
$Y = 3$	0.1	0	0.3
$Y = 4$	0	0.1	0

$\mathbb{E}[X] =$

$\mathbb{E}[Y] =$

Expectation of a Bernoulli

Recall that if $X \sim \text{Bernoulli}(p)$

$$\mathbb{P}[X = 1] = p$$

$$\mathbb{P}[X = 0] = 1 - p$$

Then: $\mathbb{E}[X] =$

Mixing Up HW

n students turn in their HW, but I accidently mix them up. I return HW to the students, so that each mixup (**permutation**) is equally likely.

What is the expected number of students who get their own HW back?

Linearity of Expectation

The definition of expectation isn't always easy to use. **Linearity** remedies this.

Theorem: Let $X_1, X_2, \ldots, X_n$ be RVs over the same probability space.
They are **not necessarily independent.** Then:

$$\mathbb{E}[X_1 + \ldots + X_n] = \mathbb{E}[X_1] + \ldots + \mathbb{E}[X_n]$$

For **constant** c, $\quad \mathbb{E}[cX_i] = c \cdot \mathbb{E}[X_i]$

Proof: Notes. Out of scope, but not a hard proof.
Maybe formally go through it next lecture.

Linearity: Example I

	$X = 1$	$X = 2$	$X = 3$
$Y = 2$	0.2	0.2	0.1
$Y = 3$	0.1	0	0.3
$Y = 4$	0	0.1	0

From previous: $\mathbb{E}[X] = 2.1$, $\mathbb{E}[Y] = 2.6$.

$\mathbb{E}[3X + 7Y] =$

Expectation of a Binomial

Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli(p) RVs.
Let $X = X_1 + \ldots + X_n$.

$$X \sim$$

What is $\mathbb{E}[X]$?

Linearity: Example II

I draw two cards from a standard deck.
What is the expected number of aces I get?

Attempt #1: Use the definition.

Linearity: Example II

Attempt #2: Use linearity of expectation.
$C_1 =$
$C_2 =$

A Note on Symmetry

C_i = indicator for the i-th card being an ace.

$$\mathbb{P}[C_i = 1] =$$

Now, imagine I draw the entire deck.

$$\mathbb{E}[C_1 + C_2 + \ldots + C_{52}] =$$

Using this, **for any** i, what is $\mathbb{E}[C_i]$?

Linearity: Mixing Up HW

(From notes.)
Same HW setup as before with n students.

S_i = indicator variable for

Summary

- Joint distribution: **multiple** RVs. Can still be defined for **non-independent** RVs.
- Ideas of independence, conditional probability **same as before**.
- **Expectation** describes the **weighted average** of a RV.
- For more complicated RVs, break down into smaller parts (e.g. **indicator variables**) and **use linearity**