RVs Continued: Joint Distribution and Intro to Expectation

CS 70, Summer 2019

Lecture 19, 7/25/19

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From Yesterday...

- RVs assign numbers to outcomes.
- Treat X = i as any ordinary event.
- Bernoulli, Binomial, Geometric, Poisson RVs.Today:
 - Joint Distributions, Independent RVs, Conditional Probability
 - Introduction to expectation and linearity of expectation

Joint Distribution

Lets you work with **multiple** random variables. No different from **intersections of events!**

RV X: takes values a in set A \mathcal{L} Defining sets of RV Y: takes values b in set B \mathcal{L} events.

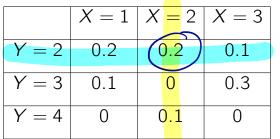
Joint Distribution: X=0 Y=bValues: $\{(a, b): a \in A, b \in B\}$ of w mma Specify the Probabilities: $P[(a,b)] = P[X=a \cap Y=b] = P[X=a, Y=b]$

Joint Distribution: Example I

$$X = 1$$
 $X = 2$
 $X = 3$
 $P[x=3, r=2]$
 $Y = 2$
 0.2
 0.1
 0.1
 $Y = 3$
 0.1
 0
 0.3
 $Y = 4$
 0
 0.1
 0

 $\mathbb{P}[(X \text{ even}) \cap (Y \text{ even})] = \mathbb{P}[X=2, Y=2] + \mathbb{P}[X=2, Y=4]$ = 0.2+0.1 = 0.3

Joint Distribution: Example II



 $\mathbb{P}[X = 2] = 0.2 + 0 + 0.1 = 0.3$ $\mathbb{P}[Y = 2] = 0.2 + 0.2 + 0.1 = 0.5$ $\mathbb{P}[(X = 2) \cap (Y = 2)] = 0.2$

Are the events X=2, Y=2independent? $(0.3)(0.5)\stackrel{?}{=}0.2$ X

Joint Distribution: Example III

$$X = 1$$
 $X = 2$ $X = 3$ $Y = 2$ 0.2 0.2 0.1 $Y = 3$ 0.1 0 0.3 $Y = 4$ 0 0.1 0

Are the events X = 1 and Y = 2 independent? P[X=1] = 0.2 + 0.1 = 0.3 $P[Y=2] = 0.5 \leftarrow Last slide.$ $P[(X=1)\cap(Y=2)] = 0.2$

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Independent Random Variables

RVs X (values in A) and Y (values in B) are **independent** if:

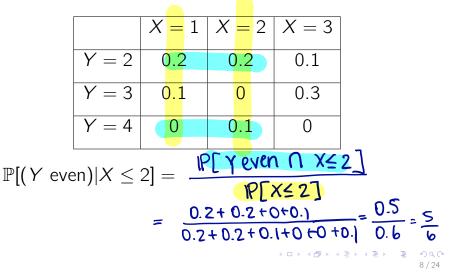
For all
$$a \in A$$
, $b \in B$:

$$\mathbb{P}[X = a, Y = b] = \mathbb{P}[X = \alpha] \mathbb{P}[Y = b]$$

Essentially the same story as ordinary events!!

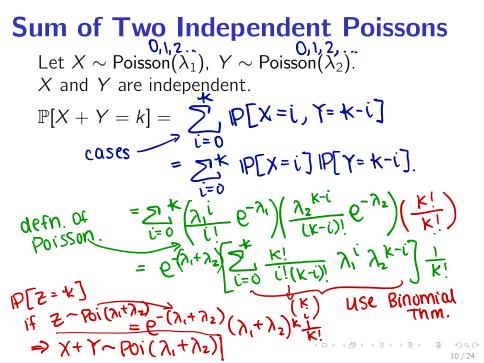
Conditional Distributions

Also the **same exact story**:



Memorylessness of Geometrics **Memoryless:** For all positive integers s, t: $\mathbb{P}[X \ge s + t | X > t] = \mathbb{P}[X \ge s]$ redundant condition. Let $X \sim \text{Geometric}(p)$. X is memoryless: $HU: \mathbb{P}[X \ge S + t | X > t] = \mathbb{P}[(X \ge S + t) \cap (X > t)]$ $= \frac{IP[X \ge Stt]}{IP[X>t]}$ $= \frac{(1-p)^{s+t-1}}{(1-p)^{t}} = (1-p)^{s-1}$ <u>RHS</u>: IP[X≥S]=(1-P)S-1 イロト イポト イラト イラト 一日

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Which building on or near campus is your "spirit building"?

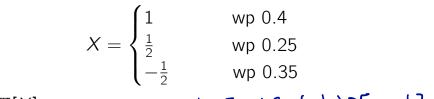
Expectation of a RV

Also called the **mean** or **average** of a RV. Let X be a RV with values in A.

Its **expectation** is defined as:

$$\mathbb{E}[X] = \sum_{\alpha \in A} \alpha \cdot \mathbb{P}[X = \alpha]$$

Expectation of a RV: Example I



 $\mathbb{E}[X] = 1 \cdot \mathbb{P}[X=1] + \pm \mathbb{P}[X=\pm] + (-\pm)\mathbb{P}[X=\pm] .$ = 1 \cdot 0.4 + \pm \cdot 0.25 + -\frac{1}{2} \cdot 0.35 = 0.4 + 0.125 - 0.175 -0.05 = 0.35

Expectation of a RV: Example III

$$X = 1$$
 $X = 2$ $X = 3$ $Y = 2$ 0.2 0.2 0.1 $Y = 3$ 0.1 0 0.3 $Y = 4$ 0 0.1 0

$$\mathbb{E}[X] = \frac{1 \cdot \mathbb{P}[X=1] + 2 \cdot \mathbb{P}[X=2] + 3 \cdot \mathbb{P}[X=3]}{= 1 \cdot (0.3) + 2 \cdot (0.3) + 3 \cdot (0.4) = 2 \cdot 1}$$
$$\mathbb{E}[Y] = 2 \cdot (0.5) + 3 \cdot (0.4) + 4 \cdot (0.1)$$
$$= 1 + 1 \cdot 2 + 0 \cdot 4 = 2 \cdot 6$$

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Expectation of a Bernoulli

Recall that if $X \sim \text{Bernoulli}(p)$

$$\mathbb{P}[X = 1] = p$$
$$\mathbb{P}[X = 0] = 1 - p$$
Then: $\mathbb{E}[X] = 0 \cdot (1 - p) + 1 \cdot p$

Mixing Up HW

n students turn in their HW, but I accidently mix them up. I return HW to the students, so that each mixup (**permutation**) is equally likely.

What is the expected number of students who get their own HW back?

casus: $0, 1, 2, \ldots, n-2, n$.

Size of prob space: n! Finding IP[#fixed -i] is hard! >> Def'n is not enough!

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Linearity of Expectation

The definition of expectation isn't always easy to use. **Linearity** remedies this.

Theorem: Let $X_1, X_2, ..., X_n$ be RVs over the same probability space. They are **not necessarily independent.** Then:

()
$$\mathbb{E}[X_1 + \ldots + X_n] = \mathbb{E}[X_1] + \ldots + \mathbb{E}[X_n]$$

(2) For constant c , $\mathbb{E}[cX_i] = c \cdot \mathbb{E}[X_i]$

Proof: Notes. Out of scope, but not a hard proof. Maybe formally go through it next lecture. Probably

Linearity: Example I

$$X = 1$$
 $X = 2$ $X = 3$ $Y = 2$ 0.2 0.2 0.1 $Y = 3$ 0.1 0 0.3 $Y = 4$ 0 0.1 0

From previous:
$$\mathbb{E}[X] = 2.1$$
, $\mathbb{E}[Y] = 2.6$.
 $\mathbb{E}[3X + 7Y] = \mathbb{E}[3X] + \mathbb{E}[7Y]$
 $= 3 \mathbb{E}[X] + 7 \mathbb{E}[Y]$
 $= 3 \cdot 2.1 + 7 \cdot 2.6$
 $= 6.3 + 18.2 = 24.5$

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Expectation of a Binomial

Let X_1, \ldots, X_n be i.i.d. Bernoulli(p) RVs. Let $X = X_1 + \ldots + X_n$. ind., (dentically distributed).

What is
$$\mathbb{E}[X]$$
?
 $\mathbb{E}[X] = \mathbb{E}[X_1 + X_2 + ... + X_n]$
 $= \mathbb{E}[X_1] + \mathbb{E}[X_2] + ... + \mathbb{E}[X_n]$
 $= \mathbb{N} \cdot \mathbb{E}[X_1] = \mathbb{N}p$
 $\stackrel{e}{=} p \text{ from}$
previous suide.

Linearity: Example II

I draw two cards from a standard deck. What is the expected number of aces I get? Attempt #1: Use the definition. $A = \# \alpha c e \alpha$. $\mathbb{E}\left[A\right] = 0 \cdot \mathbb{P}\left[A = 0\right]^{2} + [\cdot \mathbb{P}\left[A = 1\right] + 2 \cdot \mathbb{P}\left[A = 2\right]$ P[A=1] = P[notace, ace] + P[ace, not] $= (\frac{48}{52})(\frac{4}{51}) + (\frac{4}{52})(\frac{48}{51}) = \frac{384}{52 \cdot 51}$ $P[A=2] = \left(\frac{4}{52}\right)\left(\frac{3}{51}\right) = \frac{12}{52\cdot51} = \frac{8}{408}$ $IE[A] = 1 \cdot \frac{384}{52\cdot51} + 2 \cdot \frac{12}{52\cdot51} = \frac{408}{52\cdot51}$

Linearity: Example II

Attempt #2: Use linearity of expectation. $C_1 =$ indicator variable for card 1 = Ace. card 2 = Ace- 16 1 11 $C_{2} =$ $C_1 \sim Ber(\frac{1}{5}) = Ber(\frac{1}{13})$ $A = C_1 + C_2$ C2~Ber.(古) E[C门=古、E[C2]=古 $E[A] = E[C_1 + C_2] = E[C_1] + E[C_2]$ $=\frac{1}{13}+\frac{1}{13}=\frac{2}{12}$

A Note on Symmetry

 C_i = indicator for the *i*-th card being an ace.

$$\mathbb{P}[C_i=1] = \frac{1}{13}$$

Now, imagine I draw the entire deck.

 $\mathbb{E}[C_1 + C_2 + \ldots + C_{52}] = 4$ # of aces I get. = 4, always Using this, for any *i*, what is $\mathbb{E}[C_i]$? $\frac{4}{52} = \frac{1}{13}$

Linearity: Mixing Up HW

(From notes.) Same HW setup as before with *n* students. Want, Expected # of fixed Doints $S_i =$ indicator variable for student i getting their own HW. Si~Ber (1) S = # of fixed points (students who get own HW back) $S=S_1+S_2+\ldots+S_n$ 1E[S] = 1E[S1 + S2 + . + Sn] * IETS,]+ IE[S2]+. .+ IE[Sn] = $n \cdot E[S_i] = n(\frac{1}{n}) = 1$ ・ロト ・ 四 ト ・ 回 ト ・ 日 ト

Summary

- Joint distribution: multiple RVs. Can still be defined for non-independent RVs.
- Ideas of independence, conditional probability same as before.
- Expectation describes the weighted average of a RV.
- For more complicated RVs, break down into smaller parts (e.g. indicator variables) and use linearity