Lecture 2: Proofs

No, not the alcohol kind

Introduction to Proofs

What are proofs?

- Sequence of logical deductions
- Deduce new claims from already known
- Mix of English and mathematical notation

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  - Informal methods can be misleading!
- Collect thoughts into a crisp, clear argument
- Convince others that something is true

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Today: general proof techniques + examples

Direct Proof

Many theorems take the form $P \implies Q$

- eg, "n is even $\implies n^2$ is even"

Direct proofs do exactly what you would expect: suppose P is true[1] and deduce that Q is also true.

[1] if P is not true, the implication holds vacuously!

Direct Proof Example

Theorem: If $a|b$ ("a divides b") and $a|c$, $a|(b+c)$

[2] In fact, $a|(xb+yc)$ for all integers x and y!

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Proof does not specify what values a, b, and c take on — proves the statement for all a, b, and c!

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Similar method to show $a|(b-c)$[2]

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Proof(only if):

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- Suppose $9|n$, so $n = 100a + 10b + c = 9j$
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Recall: $P \implies Q \equiv (\neg Q) \implies (\neg P)$

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- $\neg Q$ might give more information than P
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Proof by contraposition is just a direct proof of the contrapositive.

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- Thus n^2 is odd

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- Take $y = \frac{x}{2}$
- Since $x > 0$, $x > \frac{x}{2} > 0$

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Sometimes called a "proof by example" (or a "proof by counterexample" for disproving a "for all")

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Theorem: Suppose we place n items into k boxes. If $n > k$, at least one box has more than one item.[3]

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- Let n_i be the number of items in box i
- Suppose that $n_i \leq 1$ for all i
- Then $n = n_1 + ... + n_k \leq 1 + ... + 1 = k$

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Proof by Contradiction

Idea: show that P being false is nonsensical

Formally: show that $\neg P$ implies something false[4]

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Why does this work?

Contrapositive of $(\neg P) \implies$ False is True $\implies P$

Intuition: $(\neg P) \implies$ False, so $\neg P$ can't be true.
But if $\neg P$ is false, P is true by definition!

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- Next time: every number has a prime factor
- Contradiction! Must be infinitely many primes

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- Since $\frac{a}{b} = \sqrt{2}$, $a^2 = 2b^2$
- a^2 even, so $a = 2k$

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- a^2 even, so $a = 2k$
- Hence $b^2 = 2k^2$, so b^2 and b even
- a and b both even, so $\frac{a}{b}$ not in lowest terms!
- Contradiction! $\sqrt{2}$ must be irrational

Break Time!

Whew, time for a 4 minute break.

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Today's discussion question:
Which is the one true kind of peanut butter: chunky or smooth?

Proof by Cases

Idea: one of these cases happens, but which one?
Prove the claim in each case.

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Why does this work?
Consider two cases C_1 and C_2. If $C_1 \implies P$ and $C_2 \implies P$, then $(C_1 \vee C_2) \implies P$!

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- Take $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$
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But which case?
Doesn't matter, but is case 2.

Proof by Cases Example 2

Theorem: Let $x, y \in \mathbb{R}$. Then $|x + y| \leq |x| + |y|$.[5]

[5] This is known as the *triangle inequality*.

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Case 2: $x \geq 0$, $y < 0$

- If $|x| \geq |y|$, $|x+y| = |x| - |y| \leq |x| + |y|$
- Else $|x+y| = |y| - |x| \leq |y| + |x| = |x| + |y|$

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Case 3: $x < 0$, $y \geq 0$

- Switch x and y to get case 2!

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Case 3: $x < 0$, $y \geq 0$

- Switch x and y to get case 2!

Case 4: $x < 0$, $y < 0$

- Negate x and y to get case 1!

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Be careful when writing proofs! Very easy to miss small errors that break everything :(

[6] In fact, if you start with a false assumption, you can prove anything. This is known as the *Principle of Explosion*.

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Consider the following "proof":
Claim: $-2 = 2$
"Proof":

- Suppose $-2 = 2$
- Square both sides to get $4 = 4$
- This is true, so we must have that $-2 = 2$

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Tried to use $P \implies \text{True}$ to conclude P.
But this implication holds even if P is false![6]

[6] In fact, if you start with a false assumption, you can prove anything. This is known as the *Principle of Explosion*.

Other Common Errors

Claim: $1 = 2$

“Proof”:

- Let x and y be integers such that $x = y$
- Then $x^2 - xy = x^2 - y^2$
- Divide by $x - y$ to get $x = x + y$
- Take $x = y = 1$ to get $1 = 2$

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Issue: $x = y$, so $x - y = 0$. Divided by zero!

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"Proof":

- We know that $-2 \leq 1$
- Square both sides to get $4 \leq 1$

Issue: squaring multiplied by -2 — flips inequality!

Tips for Proofs

Proof-writing is a skill, and can be difficult. Here are some tips on how to write your own:

- Use full English sentences for clarity.
    - Proofs should be clear enough to convince a skeptical classmate
- Use lemmas to break up a long proof.
- Develop your style through practice.
- Read other's proofs to see their style.

Fin

Next time: induction!