Expectation Continued: Tail Sum, Coupon Collector, and Functions of RVs

CS 70, Summer 2019

Lecture 20, 7/29/19

Last Time...

- **Expectation** describes the **weighted average** of a RV.
- For more complicated RVs, **use linearity**

Today:

- Proof of linearity of expectation
- The tail sum formula
- Expectations of **Geometric and Poisson**
- Expectation of a **function** of an RV

Sanity Check

Let X be a RV that takes on values in A.
Let Y be a RV that takes on values in B.
Let $c \in \mathbb{R}$ be a constant.

Both $c \cdot X$ and $X + Y$ are also RVs!

Proof of Linearity of Expectation I

Recall linearity of expectation:

$$\mathbb{E}[X_1 + \ldots + X_n] = \mathbb{E}[X_1] + \ldots + \mathbb{E}[X_n]$$

For **constant** c, $\quad \mathbb{E}[cX_i] = c \cdot \mathbb{E}[X_i]$

First, we show $\mathbb{E}[cX_i] = c \cdot \mathbb{E}[X_i]$:

Proof of Linearity of Expectation II

Next, we show $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.

Two variables to n variables?

The Tail Sum Formula

Let X be a RV with values in $\{0, 1, 2, \ldots, n\}$.
We use **"tail"** to describe $\mathbb{P}[X \geq i]$.

What does $\sum_{i=1}^{\infty} \mathbb{P}[X \geq i]$ look like?

Small example: X only takes values $\{0, 1, 2\}$:

The Tail Sum Formula

The **tail sum formula** states that:

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \mathbb{P}[X \geq i]$$

Proof: Let $p_i = \mathbb{P}[X = i]$.

Expectation of a Geometric I

Let $X \sim \text{Geometric}(p)$.

$$\mathbb{P}[X \geq i] =$$

Apply the **tail sum formula**:

Expectation of a Geometric II

Use **memorylessness**: the fact that the geometric RV **"resets"** after each trial.

Two Cases:

Expectation of a Geometric III

Lastly, an intuitive but **non-rigorous** idea.

Let X_i be an indicator variable for success in a single trial. Recall trials are **i.i.d.**

$$X_i \sim$$

$\mathbb{E}[X_1 + X_2 + \ldots + X_k] =$

Coupon Collector I

(Note 19.) I'm out collecting trading cards. There are n types total. I get a random trading card every time I buy a cereal box.

What is the **expected number of boxes** I need to buy in order to get all n trading cards?

High level picture:

Coupon Collector II

Let $X_i =$

What is the dist. of X_1?

What is the dist. of X_2?

What is the dist. of X_3?

In general, what is the dist. of X_i?

Coupon Collector III

Let $X =$

$$X =$$

$\mathbb{E}[X] =$

Aside: (Partial) Harmonic Series

Harmonic Series: $\sum_{k=1}^{\infty} \frac{1}{k}$

Approximation for $\sum_{k=1}^{n} \frac{1}{k}$ in terms of n?

Break

A Bad Harmonic Series Joke...

A countably infinite number of mathematicians walk into a bar. The first one orders a pint of beer, the second one orders a half pint, the third one orders a third of a pint, the fourth one orders a fourth of a pint, and so on.

The bartender says ...

Expectation of a Poisson I

Recall the Poisson distribution: values $0, 1, 2, \dots,$

$$\mathbb{P}[X = i] = \frac{\lambda^i}{i!} e^{-\lambda}$$

We can use the **definition** to find $\mathbb{E}[X]$!

Expectation of a Poisson II

Optional but intuitive / non-rigorous approach:

Think of a Poisson(λ) as a Bin$(n, \frac{\lambda}{n})$ distribution, taken as $n \to \infty$.

Let $X \sim \text{Bin}(n, \frac{\lambda}{n})$.

$X =$

Rest of Today: Functions of RVs!

Recall X from Lecture 19:

$$X = \begin{cases} 1 & \text{wp } 0.4 \\ \frac{1}{2} & \text{wp } 0.25 \\ -\frac{1}{2} & \text{wp } 0.35 \end{cases}$$

Refresh your memory: What is X^2?

Example: Functions of RVs

$$X^2 = \begin{cases} 1 & \text{wp } 0.4 \\ \frac{1}{4} & \text{wp } 0.6 \end{cases}$$

What is $\mathbb{E}[X^2]$?

What is $\mathbb{E}[3X^2 - 5]$?

In General: Functions of RVs

Let X be a RV with values in A.
Distribution of $f(X)$:

$\mathbb{E}[f(X)] =$

Square of a Bernoulli

Let $X \sim \text{Bernoulli}(p)$.
Write out the distribution of X.

What is X^2? $\mathbb{E}[X^2]$?

Product of RVs

Let X be a RV with values in A.
Let Y be a RV with values in B.

XY is also a RV! What is its distribution?
(Use the **joint distribution!**)

Product of Two Bernoullis

Let $X \sim \text{Bernoulli}(p_1)$, and $Y \sim \text{Bernoulli}(p_2)$.
X and Y are **independent**.

What is the distribution of XY?

What is $\mathbb{E}[XY]$?

Square of a Binomial I

Let $X \sim \text{Bin}(n, p)$.
Decompose into $X_i \sim \text{Bernoulli}(p)$.

$X =$

$\mathbb{E}[X] =$

Square of a Binomial II

Recall, $\mathbb{E}[X_i^2] = p$, and $\mathbb{E}[X_i X_j] = p^2$.

Summary

Today:

- Proof of linearity of expectation: did not use independence, but did use **joint distribution**
- Tail sum for **non-negative int.-valued** RVs!
- Coupon Collector: break problem down into a **sum of geometrics.**
- Expectation of a **function** of an RV: can apply definition and linearity of expectation (after expanding) as well!!