Expectation Continued: Tail Sum, Coupon Collector, and Functions of RVs

CS 70, Summer 2019

Lecture 20, 7/29/19

Last Time...

- **Expectation** describes the **weighted average** of a RV.
- For more complicated RVs, **use linearity**

Today:

- Proof of linearity of expectation
- The tail sum formula
- Expectations of **Geometric and Poisson**
- Expectation of a **function** of an RV

Sanity Check

Let X be a RV that takes on values in A.
Let Y be a RV that takes on values in B.
Let $c \in \mathbb{R}$ be a constant.

Both $c \cdot X$ and $X + Y$ are also RVs!

cX

Values: $\{c \cdot a, a \in A\}$

Probs:
$\mathbb{P}[cX = ca] = \mathbb{P}[X = a]$

$X+Y$

Values: $\{a+b, a \in A, b \in B\}$

Probs:
$\mathbb{P}[X+Y = a+b] = \mathbb{P}[X=a, Y=b]$

Proof of Linearity of Expectation I

Recall linearity of expectation:

$$\mathbb{E}[X_1 + \ldots + X_n] = \mathbb{E}[X_1] + \ldots + \mathbb{E}[X_n]$$

$$\text{For } \textbf{constant } c, \quad \mathbb{E}[cX_i] = c \cdot \mathbb{E}[X_i]$$

X_i values in A.

First, we show $\mathbb{E}[cX_i] = c \cdot \mathbb{E}[X_i]$:

$$\mathbb{E}[cX_i] = \sum_{a \in A} (ca)\, \mathbb{P}[X = a]$$

$$= c \underbrace{\sum_{a \in A} a \cdot \mathbb{P}[X = a]}_{\mathbb{E}[X_i]} = c\,\mathbb{E}[X_i]$$

Proof of Linearity of Expectation II

Next, we show $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.

X has values in A, Y has values in B.

$$\mathbb{E}[X+Y] = \sum_{a\in A, b\in B} (a+b)\,\mathbb{P}[X=a, Y=b]$$

$$= \sum_{a\in A, b\in B} a\cdot \mathbb{P}[X=a,Y=b] + \sum_{a\in A, b\in B} b\cdot \mathbb{P}[X=a,Y=b]$$

rewrite as $\sum_{a\in A}\sum_{b\in B}$ or $\sum_{b\in B}\sum_{a\in A}$

$$= \sum_{a\in A} a \underbrace{\sum_{b\in B} \mathbb{P}[X=a,Y=b]}_{\mathbb{P}[X=a]} + \sum_{b\in B} b \underbrace{\sum_{a\in A} \mathbb{P}[X=a,Y=b]}_{\mathbb{P}[Y=b]}$$

$$= \underbrace{\sum_{a\in A} a\cdot \mathbb{P}[X=a]}_{\mathbb{E}[X]} + \underbrace{\sum_{b\in B} b\cdot \mathbb{P}[Y=b]}_{\mathbb{E}[Y]}$$

Two variables to n variables?

Induction.

The Tail Sum Formula

Let X be a RV with values in $\{0, 1, 2, \ldots, \overset{\infty}{n}\}$. non-neg integers

We use **"tail"** to describe $\mathbb{P}[X \geq i]$.

What does $\sum_{i=1}^{\infty} \mathbb{P}[X \geq i]$ look like?

Small example: X only takes values $\{0, 1, 2\}$:

$$\sum_{i=1}^{2} \mathbb{P}[X \geq i] = \mathbb{P}[X \geq 1] + \mathbb{P}[X \geq 2]$$

$$= \mathbb{P}[X=1] + \mathbb{P}[X=2] + \mathbb{P}[X=2]$$

$$= \underbrace{0 \cdot \mathbb{P}[X=0] + 1 \cdot \mathbb{P}[X=1] + 2 \cdot \mathbb{P}[X=2]}_{\mathbb{E}[X]}$$

The Tail Sum Formula

The **tail sum formula** states that: on X non-neg. integers

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \mathbb{P}[X \geq i]$$

Proof: Let $p_i = \mathbb{P}[X = i]$.

$$\sum_{i=1}^{\infty} \mathbb{P}[X \geq i] = \mathbb{P}[X \geq 1] + \mathbb{P}[X \geq 2] + \mathbb{P}[X \geq 3] + \dots$$

$$= (p_1 + \underline{p_2} + p_3 + \dots) + (\underline{p_2} + p_3 + p_4 + \dots) + (p_3 + p_4 + \dots)$$

$$= 0 \cdot p_0 + 1 \cdot p_1 + 2 \cdot p_2 + 3 \cdot p_3 + \dots$$

$$= \mathbb{E}[X]$$

Expectation of a Geometric I

Let $X \sim \text{Geometric}(p)$.

$$\mathbb{P}[X \geq i] = (1-p)^{i-1}$$

Apply the **tail sum formula**:

$$\sum_{i=1}^{\infty} \mathbb{P}[X \geq i] = \underbrace{1 + (1-p) + (1-p)^2 + \ldots}_{\text{Geometric series}}$$

first term $\rightarrow a = 1$, $r = (1-p)$

$$= \frac{a}{1-r} = \frac{1}{1-(1-p)} = \frac{1}{p}$$

Expectation of a Geometric II

$X \sim \mathrm{Geom}(p)$

Use **memorylessness**: the fact that the geometric RV **"resets"** after each trial.

Two Cases:

start trials

p: succeed — "win in 1 day."

$1-p$: fail 1st trial. — "win in $1+\mathbb{E}[X]$ days"

failed first trial.

"reset," day 2 looks identical to day 1

$$\mathbb{E}[X] = p(1) + (1-p)\left[1 + \mathbb{E}[X]\right]$$

Solve for $\mathbb{E}[X]$.

$$p\,\mathbb{E}[X] = \cancel{p} + 1 - \cancel{p}$$

$$\mathbb{E}[X] = \frac{1}{p}$$

Expectation of a Geometric III

Lastly, an intuitive but **non-rigorous** idea.

$X \sim \mathrm{Geom}(p)$

Let X_i be an indicator variable for success in a single trial. Recall trials are **i.i.d.**

$$X_i \sim \mathrm{Ber}(p)$$

use linearity of expectation

$$\mathbb{E}[X_1 + X_2 + \ldots + X_k] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \ldots + \mathbb{E}[X_n]$$

$$= k\,\mathbb{E}[X_1]$$

$$= kp$$

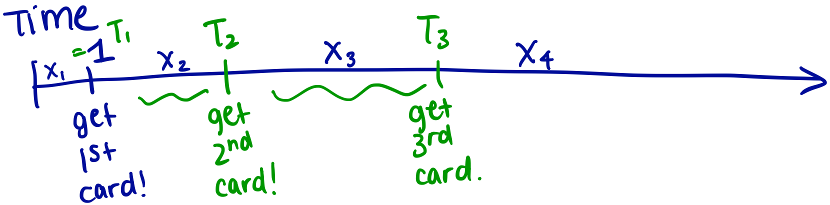
Need $k = \frac{1}{p}$ in order for $\mathbb{E}[\underbrace{X_1 + \ldots + X_k}_{\#\text{ of successes.}}] = 1$

Coupon Collector I

(Note 19.) I'm out collecting trading cards. There are n types total. I get a random trading card every time I buy a cereal box.

What is the **expected number of boxes** I need to buy in order to get all n trading cards?

High level picture:



Coupon Collector II

Let $X_i =$ time/"# boxes" between $(i-1)^{th}$ card and the i^{th}

What is the dist. of X_1? $X_1 = 1$ always. $\sim \text{Geom}(1)$

What is the dist. of X_2? $X_2 \sim \text{Geom}\left(\frac{n-1}{n}\right)$

What is the dist. of X_3? $X_3 \sim \text{Geom}\left(\frac{n-2}{n}\right)$

In general, what is the dist. of X_i?

$$X_i \sim \text{Geom}\left(\frac{n-(i-1)}{n}\right)$$

Coupon Collector III

Let X = total # boxes to get all n cards

$$X = X_1 + X_2 + \ldots + X_n$$

$$\mathbb{E}[X] = \mathbb{E}[X_1 + X_2 + \ldots + X_n]$$

linearity

$$= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \ldots + \mathbb{E}[X_n]$$

$$X_i \sim \text{Geom}\left(\frac{n-(i-1)}{n}\right)$$

$$= 1 + \frac{n}{n-1} + \frac{n}{n-2} + \ldots + \frac{n}{1}$$

$\frac{n}{n}$

$$= n \sum_{i=1}^{n} \frac{1}{i}$$

??

Aside: (Partial) Harmonic Series

Harmonic Series: $\sum_{k=1}^{\infty} \frac{1}{k}$ Diverges

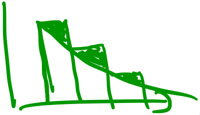
Approximation for $\sum_{k=1}^{n} \frac{1}{k}$ in terms of n?

$$\sum_{k=1}^{n} \frac{1}{k} \approx \int_1^n \frac{1}{x}\,dx$$

$$= \ln x \Big|_1^n = \ln n - \cancel{\ln 1}^{\to 0}$$

$$\approx \ln n.$$

$\Rightarrow$ coupon collector: $E[X] \approx n \log n$.



Break

A Bad Harmonic Series Joke...

A countably infinite number of mathematicians walk into a bar. The first one orders a pint of beer, the second one orders a half pint, the third one orders a third of a pint, the fourth one orders a fourth of a pint, and so on.

The bartender says ...

Expectation of a Poisson I

Recall the Poisson distribution: values $0, 1, 2, \ldots,$

$X \sim \mathrm{Poi}(\lambda)$

$$\mathbb{P}[X = i] = \frac{\lambda^i}{i!} e^{-\lambda}$$

We can use the **definition** to find $\mathbb{E}[X]$!

$$\mathbb{E}[X] = \sum_{i=0}^{\infty} \cancel{i}\left(\frac{\lambda^i}{\cancel{i!}_{(i-1)!}} e^{-\lambda}\right)$$

$$= \sum_{i=1}^{\infty} \frac{\lambda^i}{(i-1)!} e^{-\lambda}$$

$$= (e^{-\lambda})(\lambda) \underbrace{\sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!}}$$

$$= \cancel{(e^{-\lambda})}(\lambda)\cancel{e^{\lambda}} = \boxed{\lambda}$$

Taylor series for e^{λ}

$$= \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$$

Expectation of a Poisson II

Optional but intuitive / non-rigorous approach:

Think of a Poisson(λ) as a Bin($n, \frac{\lambda}{n}$) distribution, taken as $n \to \infty$.

Let $X \sim$ Bin($n, \frac{\lambda}{n}$).

$$\mathbb{E}[X] = np$$

$$= \cancel{n}\left(\frac{\lambda}{\cancel{n}}\right) = \lambda$$

Rest of Today: Functions of RVs!

Recall X from Lecture 19:

$$X = \begin{cases} 1 & \text{wp } 0.4 \\ \frac{1}{2} & \text{wp } 0.25 \\ -\frac{1}{2} & \text{wp } 0.35 \end{cases}$$

Refresh your memory: What is X^2?

$$X^2 = \begin{cases} 1 & \text{wp. } 0.4 \\ \frac{1}{4} & \text{wp } 0.25 \\ \frac{1}{4} & \text{wp } 0.35 \end{cases} = \begin{cases} 1 & \text{wp } 0.4 \\ \frac{1}{4} & \text{wp } 0.6 \end{cases}$$

Example: Functions of RVs

$$X^2 = \begin{cases} 1 & \text{wp } 0.4 \\ \frac{1}{4} & \text{wp } 0.6 \end{cases}$$

What is $\mathbb{E}[X^2]$?

$$\mathbb{E}[X^2] = 1 \cdot \mathbb{P}[X^2 = 1] + \frac{1}{4}\mathbb{P}[X^2 = \frac{1}{4}]$$

$$= 1 \cdot 0.4 + \frac{1}{4} \cdot 0.6 = 0.55$$

What is $\mathbb{E}[3X^2 - 5]$?

linearity of exp.

$$\mathbb{E}[3X^2] - \mathbb{E}[5] = 3\,\mathbb{E}[X^2] - 5$$

$$= 3(0.55) - 5$$

In General: Functions of RVs

Let X be a RV with values in A.
Distribution of $f(X)$:

$$f(X) = \begin{cases} \vdots & \\ f(a) & \text{wp } \mathbb{P}[X=a] \\ \vdots & \end{cases}$$

$$\mathbb{E}[f(X)] = \sum_{a \in A} f(a)\,\mathbb{P}[X=a]$$

Square of a Bernoulli

Let $X \sim \text{Bernoulli}(p)$.
Write out the distribution of X.

$$X = \begin{cases} 1 & \text{wp } p \\ 0 & \text{wp } 1-p \end{cases}$$

What is X^2? $\mathbb{E}[X^2]$?

$$X^2 = \begin{cases} 1 & \text{wp } p \\ 0 & \text{wp } 1-p \end{cases}$$

$$\mathbb{E}[X^2] = p.$$

Product of RVs Exercise

Let X be a RV with values in A.
Let Y be a RV with values in B.

XY is also a RV! What is its distribution?
(Use the **joint distribution!**)

Product of Two Bernoullis Exercise

Let $X \sim \text{Bernoulli}(p_1)$, and $Y \sim \text{Bernoulli}(p_2)$.
X and Y are **independent**.

What is the distribution of XY?

What is $\mathbb{E}[XY]$?

Square of a Binomial I

Let $X \sim \text{Bin}(n, p)$.
Decompose into $X_i \sim \text{Bernoulli}(p)$.

$$X = X_1 + X_2 + \ldots + X_n$$

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[X_1 + X_2 + \ldots + X_n] \\ &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \ldots + \mathbb{E}[X_n] \\ &= np.\end{aligned}$$

Square of a Binomial II

Recall, $\mathbb{E}[X_i^2] = \cancel{1}\, p$, and $\mathbb{E}[X_i X_j] = p^2$.

$$\mathbb{E}[X^2] = \mathbb{E}[(X_1 + X_2 + \dots + X_n)^2]$$

$$= \mathbb{E}\Big[\underbrace{(X_1^2 + X_2^2 + \dots + X_n^2)}_{\substack{\text{square terms} \\ n \text{ of them}}} + \underbrace{(X_1X_2 + X_1X_3 + \dots)}_{\substack{\text{cross terms} \\ n(n-1) \text{ of them.}}}\Big]$$

$$= \mathbb{E}[\text{square terms}] + \mathbb{E}[\text{cross terms}]$$

$$= n\,\mathbb{E}[X_1^2] + n(n-1)\,\mathbb{E}[X_1X_2]$$

$$X_1^2 = \begin{cases} 1 & \text{wp } p \\ 0 & 1-p \end{cases} \qquad X_1X_2 = \begin{cases} 1 & \text{wp } p^2 \\ 0 & 1-p^2 \end{cases}$$

$$\mathbb{E}[X_1^2] = p \qquad \mathbb{E}[X_1X_2] = p^2$$

$$= np + n(n-1)p^2$$

Summary

Today:

- Proof of linearity of expectation: did not use independence, but did use **joint distribution**
- Tail sum for **non-negative int.-valued** RVs!
- Coupon Collector: break problem down into a **sum of geometrics.**
- Expectation of a **function** of an RV: can apply definition and linearity of expectation (after expanding) as well!!