# Variance

CS 70, Summer 2019

Lecture 21, 7/30/19



## Two Games

**Game 1:** Flip a fair coin 10 times. For each Head, you win 100. For each Tail, you lose 100.

$\underbrace{\text{Expected Winnings}}_{F_i}$ on Flip $i$:

$$\mathbb{E}[F_i] = 100\left(\tfrac{1}{2}\right) + (-100)\left(\tfrac{1}{2}\right) = \underline{0}$$

$\underbrace{\text{Expected Winnings After 10 Flips}}_{F}$:

$$\mathbb{E}[F] = \sum_{i=1}^{10} \mathbb{E}[F_i] = 0$$



## Two Games

**Game 2:** Flip a fair coin 10 times. For each Head, you win 10000. For each Tail, you lose 10000.

$\underbrace{\text{Expected Winnings on Flip } i}_{F_i}$:

$$\mathbb{E}[F_i] = 0 = \tfrac{1}{2}(10000) + \tfrac{1}{2}(-10000)$$

$\underbrace{\text{Expected Winnings After 10 Flips}}_{\mathbb{E}[F] = 0}$:

**Q:** Which game would you rather play?



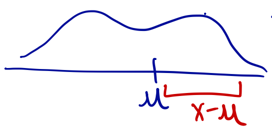
## Definition of Variance

The key difference is the **variance**.

Variance is the **expected "distance" to mean.**

Let $X$ be a RV with $\mathbb{E}[X] = \mu$. Then:

$$\text{Var}(X) = \underbrace{\mathbb{E}\left[(X-\mu)^2\right]}_{\text{variance is always non-neg}}$$




Std Dev:

$$\sigma(X) = \sqrt{\text{Var}(X)}$$



## Alternate Definition

We can use **linearity of expectation** to get an alternate form that is often **easier to apply**.

$$\text{Var}(X) = \mathbb{E}[X^2] - \mu^2$$

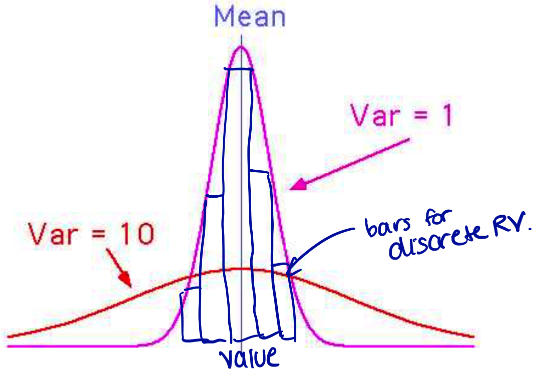
$$\mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2 - 2\mu X + \mu^2]$$

$$\xrightarrow{\text{linearity}} = \mathbb{E}[X^2] - 2\mu \underbrace{\mathbb{E}[X]}_{\mu} + \mu^2$$

$$= \mathbb{E}[X^2] - \mu^2$$



## Variance: A Visual



## Variance of a Bernoulli

Let $X \sim$ Bernoulli($p$).
Then $\mathbb{E}[X] = p$

What is $X^2$? $\mathbb{E}[X^2]$?

$$X^2 = \begin{cases} 1 & \text{wp } p \\ 0 & \text{wp } 1-p \end{cases}$$

$$\mathbb{E}[X^2] = 1 \cdot p + 0 \cdot (1-p) = p.$$

$$\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \underbrace{p} - (p)^2 = p(1-p)$$

$X \sim Ber(p)$, $Y \sim Ber(1-p)$, $Var(X) = Var(Y)$



## Variance of a Dice Roll

What is the variance of a single 6-sided dice roll?

$R =$ value of a dice roll. $\{1,2,3,4,5,6\}$

What is $R^2$?

$$R^2 = \begin{cases} 1 & \text{wp } \frac{1}{6} \\ 4 & \text{"} \ \frac{1}{6} \\ 9 & \text{"} \\ 16 & \vdots \\ 25 & \vdots \\ 36 & \end{cases}$$



## Variance of a Dice Roll

$$\mathbb{E}[R^2] = \frac{1}{6}[\underbrace{1+4+9}_{10}+\underbrace{16}_{20}+25+36]$$

$$= \frac{1}{6}(91)$$

$$\text{Var}(R) = \mathbb{E}[R^2] - (\mathbb{E}[R])^2$$

$$= \frac{91}{6} - \left(\frac{7}{2}\right)^2 =$$

(Notes.)



## Variance of a Geometric

**Know the variance;** proof optional, but good practice with **manipulating RVs**.

Let $X \sim$ Geometric($p$).
Strategy: Nice expression for $p \cdot \mathbb{E}[X^2]$

$$\mathbb{E}[X^2] = 1 \cdot p + 4(1-p)p + 9(1-p)^2 p + \ldots$$

$$-(1-p)\mathbb{E}[X^2] = -[1 \cdot (1-p)p + 4(1-p)^2 p]$$

subtract from both sides.

$$p\mathbb{E}[X^2] = 1 \cdot p + 3(1-p)p + 5(1-p)^2 p + \ldots$$

$$= (2 \cdot p + 4(1-p)p + 6(1-p)^2 p + \ldots) + (-p - (1-p)p - (1-p)^2 p - \ldots)$$



## Variance of a Geometric II

From the distribution of $X$, we know:

① $$\sum_{i=1}^{\infty} \mathbb{P}[X=i] = p + (1-p)p + (1-p)^2 p + \ldots = 1$$

From $\mathbb{E}[X]$, we know:

$$\mathbb{E}[X] = 1 \cdot p + 2 \cdot (1-p)p + 3(1-p)^2 p + \ldots$$

② $$= \frac{1}{p}$$

$$\underbrace{(2 \cdot p + 4(1-p)p + 6(1-p)^2 p + \ldots)}_{2 \cdot ②} + \underbrace{(-p - (1-p)p - (1-p)^2 p - \ldots)}_{-①}$$

$$p\mathbb{E}[X^2] = 2\mathbb{E}[X] - 1$$ solve for $\mathbb{E}[X^2]$

$$\mathbb{E}[X^2] = \frac{2-p}{p^2}$$



## Variance of a Geometric III

Recall $\mathbb{E}[X] = \frac{1}{p}$.

$$Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \frac{2-p}{p^2} - \frac{1}{p^2} = \boxed{\frac{1-p}{p^2}}$$

## Variance of a Poisson

Same: **know the variance;** proof optional, but good practice with **functions of RVs**.

Let $X \sim \text{Poisson}(\lambda)$.
Strategy: Compute $\mathbb{E}[X(X-1)]$.

$$\mathbb{E}[X(X-1)] \overset{\text{(def)}}{=} \sum_{i=0}^{\infty} i(\cancel{i-1}) \cdot \overbrace{\frac{\lambda^i}{\cancel{i!}\;(i-2)!} \cdot e^{-\lambda}}^{\mathbb{P}[X=i]}$$

$$= e^{-\lambda} \sum_{i=2}^{\infty} \frac{\lambda^i}{(i-2)!}$$

$$= e^{-\lambda} \lambda^2 \sum_{i=2}^{\infty} \frac{\lambda^{i-2}}{(i-2)!}$$

Taylor series for $e^{\lambda}$

$$= \cancel{e^{-\lambda}}(\lambda^2)(\cancel{e^{\lambda}}) = \lambda^2$$



## Variance of a Poisson II

Use $\mathbb{E}[X(X-1)]$ to compute $\text{Var}(X)$.

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \underbrace{\mathbb{E}[X(X-1)]}_{\text{last slide}} + \underbrace{\mathbb{E}[X]}_{\text{yesterday}} - (\mathbb{E}[X])^2$$

$$= \cancel{\lambda^2} + \lambda - \cancel{(\lambda)^2} = \lambda$$



## Break

Would you rather only wear sweatpants for the rest of your life, or never get to wear sweatpants ever again?



## Properties of Variance I: Scale

Let $X$ be a RV, and let $c \in \mathbb{R}$ be a constant.
Let $\mathbb{E}[X] = \mu$.

$\text{Var}(cX) = c^2 \cdot \text{Var}(X)$

$$\text{Var}(cX) = \mathbb{E}[(cX)^2] - (\mathbb{E}[cX])^2$$

lin.

$$= \mathbb{E}[c^2X^2] - (c\,\mathbb{E}[X])^2$$

$$= c^2\,\mathbb{E}[X^2] - c^2(\mathbb{E}[X])^2$$

$$= c^2\,\text{Var}(X)$$



## Properties of Variance II: Shift

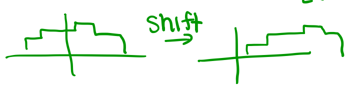
Let $X$ be a RV, and let $c \in \mathbb{R}$ be a constant.
Let $E[X] = \mu$.
Then, let $\mu' = \mathbb{E}[X + c] = \mu + c$

$\text{Var}(X + c) = \text{Var}(X)$

$$\text{Var}(X+c) = \mathbb{E}[((X+c) - \mu')^2]$$

$$= \mathbb{E}[(X + \cancel{c} - \mu - \cancel{c})^2]$$

$$= \mathbb{E}[(X-\mu)^2] = \text{Var}(X)$$






## Example: Shift It!

Consider the following RV:

$$X = \begin{cases} 1 & \text{w.p. } 0.4 \\ 3 & \text{w.p. } 0.2 \\ 5 & \text{w.p. } 0.4 \end{cases}$$

$= \text{Var}(X-3)$

What is $\text{Var}(X)$? Shift it!

$$X-3 = \begin{cases} -2 & \text{wp. } 0.4 \\ 0 & \text{wp } 0.2 \\ 2 & \text{wp } 0.4 \end{cases}$$

$$(X-3)^2 = \begin{cases} 4 & \text{wp } 0.8 \\ 0 & \text{wp } 0.2 \end{cases}$$

$\mathbb{E}[(X-3)^2] = 4 \cdot 0.8 + 0 \cdot 0.2 = 3.2$

$\mathbb{E}[X-3] = 0$

$\text{Var}[(X-3)] = 3.2$

## Sum of Independent RVs

Let $X_1, \ldots, X_n$ be independent RVs. Then:

$$\mathrm{Var}(X_1 + \ldots + X_n) = \mathrm{Var}(X_1) + \ldots + \mathrm{Var}(X_n)$$

Proof: Tomorrow!

**Today: Focus on applications.**



## Variance of a Binomial

Let $X \sim \mathrm{Bin}(n, p)$. Then,

$$X = X_1 + X_2 + \ldots + X_n$$

Here, $X_i \sim \mathrm{Ber}(p)$ $\boxed{X_i \text{ iid.}}$

$$\mathrm{Var}(X) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \ldots + \mathrm{Var}(X_n)$$
$$= n \cdot \mathrm{Var}(X_1)$$
$$= \boxed{n\,p(1-p)}$$

$X \sim \mathrm{Bin}(n, p)$
$Y \sim \mathrm{Bin}(n, 1-p)$

$\mathrm{Var}(X) = \mathrm{Var}(Y)$



## Sum of Dependent RVs

Main strategy: **linearity of expectation** and **indicator variables**

**Useful Fact:**

$$(X_1 + X_2 + \ldots + X_n)^2 =$$
$$(X_1 + X_2 + \ldots + X_n)(X_1 + X_2 + \ldots + X_n)$$
$$= \underbrace{(X_1^2 + X_2^2 + \ldots + X_n^2)}_{n} + \underbrace{(X_1X_2 + X_1X_3 + \ldots + X_{n-1}X_n)}_{n(n-1)}$$
$$= \sum_{i=1}^{n} X_i^2 + \sum_{i \neq j} X_i X_j$$

alternate: $2\sum_{i<j} X_i X_j$



## HW Mixups (Fixed Points)

(In notes.) $n$ students hand in HW. I mix up their HW randomly and return it, so that every possible mixup is **equally likely**.

Let $S = \#$ of students who get their own HW.

Last time: defined $S_i =$ indicator for student $i$ getting own HW.

$$S_i \sim \mathrm{Ber}\left(\tfrac{1}{n}\right)$$

Using **linearity of expectation**:

$$\mathbb{E}[S] = \mathbb{E}[S_1 + S_2 + \ldots + S_n] = \underbrace{\mathbb{E}[S_1]} + \mathbb{E}[S_2] + \ldots + \mathbb{E}[S_n]$$
$$= \frac{1}{n}(n) = 1$$



## HW Mixups II

Using our **useful fact**:

$$\mathbb{E}[S^2] = \mathbb{E}[(S_1 + S_2 + \ldots + S_n)^2]$$
$$= \mathbb{E}\left[\sum_{i=1}^{n} S_i^2 + \sum_{i \neq j} S_i S_j\right]$$

linearity:

$$= \mathbb{E}\left[\sum_{i=1}^{n} S_i^2\right] + \mathbb{E}\left[\sum_{i \neq j} S_i S_j\right]$$
$$= n \cdot \underbrace{\mathbb{E}[S_1^2]}_{?} + n(n-1)\underbrace{\mathbb{E}[S_1 S_2]}_{?}$$



## HW Mixups III

What is $S_i^2$? $\mathbb{E}[S_i^2]$?

$$S_1^2 = \begin{cases} 1 & \text{wp } \frac{1}{n} \\ 0 & \text{wp } 1 - \frac{1}{n} \end{cases} \qquad \mathbb{E}[S_1^2] = \frac{1}{n}$$

For $i \neq j$, what is $S_i S_j$? $\mathbb{E}[S_i S_j]$?

$$S_1 S_2 = \begin{cases} 1 \\ 0 \end{cases}$$

$\Rightarrow$ both $S_1, S_2 = 1$
$\Rightarrow$ both student 1, student 2 get own HW.

wp $\frac{1}{n}\left(\frac{1}{n-1}\right)$

$$\mathbb{E}[S_1 S_2] = \frac{1}{n}\left(\frac{1}{n-1}\right)$$

## HW Mixups IV

Put it all together to compute Var($X$).

$$\mathrm{Var}(X) = \underbrace{\mathbb{E}[X^2]} - (\mathbb{E}[X])^2$$

$$= n\,\mathbb{E}[S_1^2] + n(n-1)\,\mathbb{E}[S_1 S_2] - (1)^2$$

$$= \cancel{n}\left(\frac{1}{\cancel{n}}\right) + \cancel{n(n-1)}\,\frac{1}{\cancel{n(n-1)}} - 1$$

$$= 1$$



## Summary

**Today:**

- Variance measures how far you **deviate from mean**
- Variance is additive for **independent RVs**; proof to come tomorrow
- Use **linearity of expectation** and **indicator variables**