Variance

CS 70, Summer 2019

Lecture 21, 7/30/19

Two Games

Game 1: Flip a fair coin 10 times. For each Head, you win 100. For each Tail, you lose 100.

$\underbrace{\text{Expected Winnings}}_{F_i}$ on Flip i:

$$\mathbb{E}[F_i] = 100\left(\tfrac{1}{2}\right) + (-100)\left(\tfrac{1}{2}\right) = \underline{0}$$

$\underbrace{\text{Expected Winnings After 10 Flips}}_{F}$:

$$\mathbb{E}[F] = \sum_{i=1}^{10} \mathbb{E}[F_i] = 0$$

Two Games

Game 2: Flip a fair coin 10 times. For each Head, you win 10000. For each Tail, you lose 10000.

$\underbrace{\text{Expected Winnings on Flip } i}_{F_i}$:

$$\mathbb{E}[F_i] = 0 = \tfrac{1}{2}(10000) + \tfrac{1}{2}(-10000)$$

$\underbrace{\text{Expected Winnings After 10 Flips}}_{\mathbb{E}[F] = 0}$:

Q: Which game would you rather play?

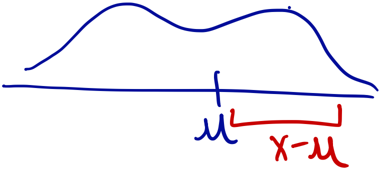
Definition of Variance

The key difference is the **variance**.

Variance is the **expected "distance" to mean.**

Let X be a RV with $\mathbb{E}[X] = \mu$. Then:

$$\mathrm{Var}(X) = \underbrace{\mathbb{E}\left[(X-\mu)^2\right]}_{\text{variance is always non-neg}}$$



Std Dev:

$$\sigma(X) = \sqrt{\mathrm{Var}(X)}$$

Alternate Definition

We can use **linearity of expectation** to get an alternate form that is often **easier to apply**.

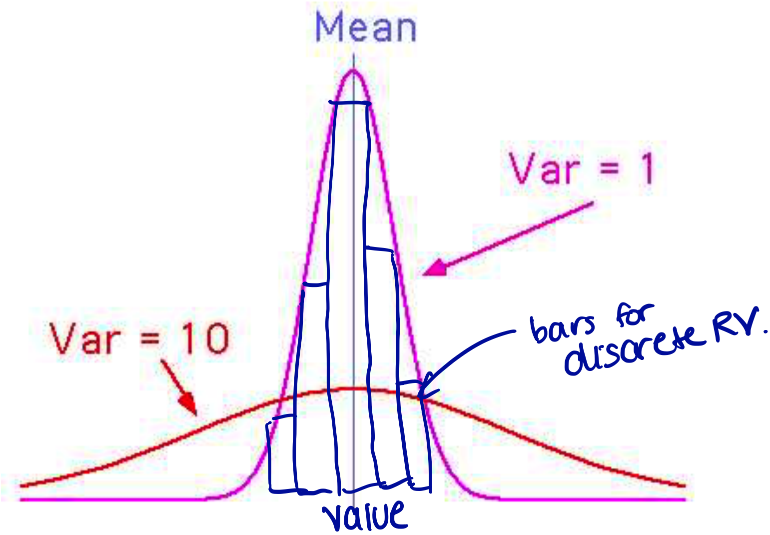
$$\text{Var}(X) = \mathbb{E}[X^2] - \mu^2$$

$$\mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2 - 2\mu X + \mu^2]$$

$$\xrightarrow{\text{linearity}} \quad = \mathbb{E}[X^2] - 2\mu \underbrace{\mathbb{E}[X]}_{\mu} + \mu^2$$

$$= \mathbb{E}[X^2] - \mu^2$$

Variance: A Visual



Variance of a Bernoulli

Let $X \sim \text{Bernoulli}(p)$.
Then $\mathbb{E}[X] = p$

What is X^2? $\mathbb{E}[X^2]$?

$$X^2 = \begin{cases} 1 & \text{wp } p \\ 0 & \text{wp } 1-p \end{cases}$$

$$\mathbb{E}[X^2] = 1 \cdot p + 0 \cdot (1-p) = p.$$

$$\text{Var}[X] = \underbrace{\mathbb{E}[X^2]}_{p} - (\mathbb{E}[X])^2$$

$$= p - (p)^2 = p(1-p)$$

$X \sim Ber(p)$, $Y \sim Ber(1-p)$
$Var(X) = Var(Y)$

Variance of a Dice Roll

What is the variance of a single 6-sided dice roll?

$R =$ value of a dice roll. $\{1,2,3,4,5,6\}$

What is R^2?

$$R^2 = \begin{cases} 1 & \text{wp } \frac{1}{6} \\ 4 & \text{"} \ \frac{1}{6} \\ 9 & \vdots \quad \vdots \\ 16 & \\ 25 & \\ 36 & \end{cases}$$

Variance of a Dice Roll

$$\mathbb{E}[R^2] = \frac{1}{6}\left[1 + 4 + 9 + 16 + 25 + 36\right]$$

10 20

$$= \frac{1}{6}(91)$$

$$\mathrm{Var}(R) = \mathbb{E}[R^2] - (\mathbb{E}[R])^2$$

$$= \frac{91}{6} - \left(\frac{7}{2}\right)^2 =$$

(Notes.)

Variance of a Geometric

Know the variance; proof optional, but good practice with **manipulating RVs**.

Let $X \sim \text{Geometric}(p)$.
Strategy: Nice expression for $p \cdot \mathbb{E}[X^2]$

$\mathbb{E}[X^2] = 1 \cdot p + 4(1-p)p + 9(1-p)^2 p + \ldots$

$-\left[(1-p)\mathbb{E}[X^2]\right] = \qquad - \left[1 \cdot (1-p)p + 4(1-p)^2 p\right]$

subtract from both sides.

$p\,\mathbb{E}[X^2] = 1 \cdot p + 3(1-p)p + 5(1-p)^2 p + \ldots$

$= (2 \cdot p + 4(1-p)p + 6(1-p)^2 p + \ldots) + (-p - (1-p)p - (1-p)^2 p - \ldots)$

Variance of a Geometric II

From the distribution of X, we know:

① $\sum_{i=1}^{\infty} \mathbb{P}[X=i] = p + (1-p)p + (1-p)^2 p + \dots = 1$

From $\mathbb{E}[X]$, we know:

$\mathbb{E}[X] = 1 \cdot p + 2 \cdot (1-p)p + 3(1-p)^2 p + \dots$

② $= \frac{1}{p}$

$2 \cdot ②$ $\quad$ $-①$

$(2 \cdot p + 4(1-p)p + 6(1-p)^2 p + \dots) + (-p - (1-p)p - (1-p)^2 p - \dots)$

$p\mathbb{E}[X^2] = 2\mathbb{E}[X] - 1$ Solve for $\mathbb{E}[X^2]$

$\mathbb{E}[X^2] = \frac{2-p}{p^2}$

Variance of a Geometric III

Recall $\mathbb{E}[X] = \frac{1}{p}$.

$$\mathrm{Var}(X) = \underbrace{\mathbb{E}[X^2]}_{} - \underbrace{\left(\mathbb{E}[X]\right)^2}_{}$$

$$= \frac{2-p}{p^2} - \frac{1}{p^2} = \boxed{\frac{1-p}{p^2}}$$

Variance of a Poisson

Same: **know the variance;** proof optional, but good practice with **functions of RVs**.

Let $X \sim \text{Poisson}(\lambda)$.
Strategy: Compute $\mathbb{E}[X(X-1)]$.

$$\mathbb{E}[X(X-1)] \overset{(\text{def})}{=} \sum_{i=0}^{\infty} \cancel{i(i-1)} \cdot \underbrace{\frac{\lambda^i}{\cancel{i!}_{(i-2)!}} \cdot e^{-\lambda}}_{\mathbb{P}[X=i]}$$

$$= e^{-\lambda} \sum_{i=2}^{\infty} \frac{\lambda^i}{(i-2)!}$$

$$= e^{-\lambda} \lambda^2 \sum_{i=2}^{\infty} \frac{\lambda^{i-2}}{(i-2)!}$$

Taylor series for e^{λ}

$$= \cancel{e^{-\lambda}} (\lambda^2)(\cancel{e^{\lambda}}) = \lambda^2$$

Variance of a Poisson II

Use $\mathbb{E}[X(X-1)]$ to compute $\mathrm{Var}(X)$.

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \underbrace{\mathbb{E}[X(X-1)]}_{\text{last slide}} + \underbrace{\mathbb{E}[X]}_{\text{yesterday}} - (\mathbb{E}[X])^2$$

$$= \cancel{\lambda^2} + \lambda - \cancel{(\lambda)^2} = \lambda$$

Break

Would you rather only wear sweatpants for the rest of your life, or never get to wear sweatpants ever again?

Properties of Variance I: Scale

Let X be a RV, and let $c \in \mathbb{R}$ be a constant.
Let $\mathbb{E}[X] = \mu$.

$\text{Var}(cX) =$ $c^2 \cdot \text{Var}(X)$

$$\text{Var}(cX) = \mathbb{E}[(cX)^2] - (\mathbb{E}[cX])^2$$

lin.

$$= \mathbb{E}[c^2 X^2] - (c\,\mathbb{E}[X])^2$$

$$= c^2\,\mathbb{E}[X^2] - c^2(\mathbb{E}[X])^2$$

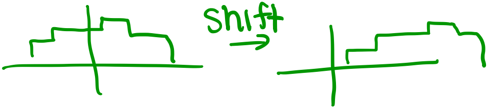
$$= c^2\,\text{Var}(X)$$

Properties of Variance II: Shift

Let X be a RV, and let $c \in \mathbb{R}$ be a constant.
Let $E[X] = \mu$.
Then, let $\mu' = \mathbb{E}[X + c] = \mu + c$

$\text{Var}(X + c) = \text{Var}(X)$

$$\text{Var}(X+c) = \mathbb{E}[((X+c) - \mu')^2]$$
$$= \mathbb{E}[(X + \cancel{c} - \mu - \cancel{c})^2]$$
$$= \mathbb{E}[(X - \mu)^2] = \text{Var}(X)$$



Example: Shift It!

Consider the following RV:

$$X = \begin{cases} 1 & \text{w.p. } 0.4 \\ 3 & \text{w.p. } 0.2 \\ 5 & \text{w.p. } 0.4 \end{cases}$$

$= \mathrm{Var}(X-3)$

What is $\mathrm{Var}(X)$? Shift it!

$$X-3 = \begin{cases} -2 & \text{wp. } 0.4 \\ 0 & \text{wp } 0.2 \\ 2 & \text{wp } 0.4 \end{cases}$$

$$(X-3)^2 = \begin{cases} 4 & \text{wp } 0.8 \\ 0 & \text{wp } 0.2 \end{cases}$$

$$\mathbb{E}[(X-3)^2] = 4 \cdot 0.8 + 0 \cdot 0.2 = 3.2$$

$$\mathbb{E}[X-3] = 0$$

$$\mathrm{Var}((X-3)) = 3.2$$

Sum of Independent RVs

Let $X_1, \ldots, X_n$ be independent RVs. Then:

$$\mathrm{Var}(X_1 + \ldots + X_n) = \mathrm{Var}(X_1) + \ldots + \mathrm{Var}(X_n)$$

Proof: Tomorrow!

Today: Focus on applications.

Variance of a Binomial

Let $X \sim \text{Bin}(n, p)$. Then,

$$X = X_1 + X_2 + \ldots + X_n$$

Here, $X_i \sim$ Ber(p) $\boxed{X_i \text{ iid.}}$

$$\text{Var}(X) = \text{Var}(X_1) + \text{Var}(X_2) + \ldots + \text{Var}(X_n)$$

$$= n \cdot \text{Var}(X_1)$$

$$= \boxed{np(1-p)}$$

$X \sim \text{Bin}(n, p)$

$Y \sim \text{Bin}(n, 1-p)$

$\text{Var}(X) = \text{Var}(Y)$

Sum of Dependent RVs

Main strategy: **linearity of expectation** and **indicator variables**

Useful Fact:

$$(X_1 + X_2 + \ldots + X_n)^2 =$$

$$(X_1 + X_2 + \ldots + X_n)(X_1 + X_2 + \ldots + X_n)$$

$$= \underbrace{(X_1^2 + X_2^2 + \ldots + X_n^2)}_{n} + \underbrace{(X_1X_2 + X_1X_3 + \ldots + X_{n-1}X_n)}_{n(n-1)}$$

$$= \sum_{i=1}^{n} X_i^2 + \sum_{i \neq j} X_iX_j$$

alternate:

$$2\sum_{i<j} X_iX_j$$

HW Mixups (Fixed Points)

(In notes.) n students hand in HW. I mix up their HW randomly and return it, so that every possible mixup is **equally likely**.

Let $S = \#$ of students who get their own HW.

Last time: defined $S_i =$ indicator for student i getting own HW.

$$S_i \sim \text{Ber}\left(\frac{1}{n}\right)$$

Using **linearity of expectation**:

$$\mathbb{E}[S] = \mathbb{E}[S_1 + S_2 + \dots + S_n] = \underbrace{\mathbb{E}[S_1]} + \mathbb{E}[S_2] + \dots + \mathbb{E}[S_n]$$

$$= \frac{1}{n}(n) = 1$$

HW Mixups II

Using our **useful fact**:

$$\mathbb{E}[S^2] = \mathbb{E}[(S_1 + S_2 + \ldots + S_n)^2]$$

$$= \mathbb{E}\left[\sum_{i=1}^{n} S_i^2 + \sum_{i \neq j} S_i S_j\right]$$

linearity:

$$= \mathbb{E}\left[\sum_{i=1}^{n} S_i^2\right] + \mathbb{E}\left[\sum_{i \neq j} S_i S_j\right]$$

$$= n \cdot \underbrace{\mathbb{E}[S_1^2]}_{?} + n(n-1)\underbrace{\mathbb{E}[S_1 S_2]}_{?}$$

HW Mixups III

What is S_i^2? $\mathbb{E}[S_i^2]$?

$$S_1^2 = \begin{cases} 1 & \text{wp } \frac{1}{n} \\ 0 & \text{wp } 1-\frac{1}{n} \end{cases}$$

$$\mathbb{E}[S_1^2] = \frac{1}{n}$$

For $i \neq j$, what is $S_i S_j$? $\mathbb{E}[S_i S_j]$?

$$S_1 S_2 = \begin{cases} 1 \\ 0 \end{cases}$$

$\Rightarrow$ both $S_1, S_2 = 1$

$\Rightarrow$ both student 1, student 2 get own HW.

wp $\frac{1}{n}\left(\frac{1}{n-1}\right)$

$$\mathbb{E}[S_1 S_2] = \frac{1}{n}\left(\frac{1}{n-1}\right)$$

HW Mixups IV

Put it all together to compute Var(X).

$$\mathrm{Var}(X) = \underbrace{\mathbb{E}[X^2]} - (\mathbb{E}[X])^2$$

$$= n\,\mathbb{E}[S_1^2] + n(n-1)\,\mathbb{E}[S_1 S_2] - (1)^2$$

$$= \cancel{n}\left(\frac{1}{\cancel{n}}\right) + \cancel{n(n-1)}\,\frac{1}{\cancel{n(n-1)}} - 1$$

$$= 1$$

Summary

Today:

- Variance measures how far you **deviate from mean**
- Variance is additive for **independent RVs**; proof to come tomorrow
- Use **linearity of expectation** and **indicator variables**