Intro to Continuous RVs

CS 70, Summer 2019

Lecture 24, 8/5/19

So Far: The PMF

Every discrete distribution we've seen so far has a **probability mass function**.

For RV X:

$$\mathrm{PMF}_X(a) = \mathbb{P}[X = a]$$

a can be any real number.
PMF_X is nonzero on... all values that X takes

Example:
Let $X \sim \mathrm{Geometric}(p)$. Then, $\mathrm{PMF}_X(a)$ is:

$$\mathrm{PMF}_X(a) = \begin{cases} 0 & \text{if } a \text{ is not positive integer} \\ (1-p)^{a-1}p & \text{else} \end{cases}$$

Uniform Over $[0, 1]$

What if X takes **uncountably** many values?

Example: I want X to be uniform over $[0, 1]$.
"Each $a \in [0, 1]$ equally likely."

What is $\mathbb{P}[X = 0]$? 0. one out of uncountably many #s

What is $\mathbb{P}[X = 0.5]$? 0

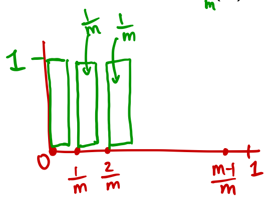
(*) What is $\mathbb{P}[a \le X \le b]$, for $0 \le a \le b \le 1$? $b-a$ probability inside proportional to length

A Discrete Approximation

Let's "approximate" X, a uniform RV over $[0, 1]$:

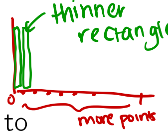
Choose integer m. Take m points "uniformly" in [0,1]
X_m take values: $0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{m-1}{m}$
Each value has probability: $\frac{1}{m}$

$$\mathrm{PMF}_{X_m}(a) = \begin{cases} 0 & \text{if } a \neq \frac{i}{m}, \text{ for } i = 0, 1, \dots, m-1 \\ \frac{1}{m} & \text{else} \end{cases}$$



Area of rectangle = height × base $= 1 \times \frac{1}{m} = \frac{1}{m}$
⇓
probability mass

A Discrete Approximation



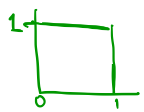
What happens as we push $m \to \infty$?

$\mathbb{P}[a \le X_m \le b]$ gets closer and closer to $\hookrightarrow b-a$

Width of each rectangle approaches 0.
In calculus, an **infinitesimally small width** is also known as: dx (*) similar to Riemann sum approx to integral.

If we attempt to write a PMF for X.

$$\mathrm{PMF}_X(x) = \begin{cases} \underbrace{1}_{\text{height}} \cdot \underbrace{dx}_{\text{width}} & \text{for all } x \in [0,1] \\ 0 & \text{o.w.} \end{cases}$$



The PDF / "Density"

Drop the dx!

The **probability density function** (PDF) for X is a function $f : \mathbb{R} \to \mathbb{R}$ such that:

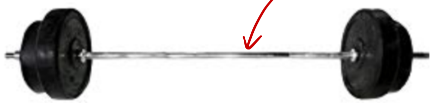
1. $f(x) \ge 0$ for all $x \in \mathbb{R}$.
   Discrete Analogue: all probabilities are non-neg.
2. $\int_{-\infty}^{\infty} f(x)dx = 1$.
   Discrete Analogue: all probabilities sum to 1.

$\mathbb{P}[a \le X \le b] = \int_a^b \underbrace{f(x)}_{\text{height}} \underbrace{dx}_{\text{width}}$
Discrete Analogue: $\sum_{i=a}^{b} \mathbb{P}[X = i]$

PMF vs. PDF: A Visual

PMF

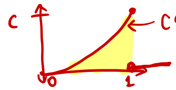
massless



PDF

Fun With PDF I

Let X be a RV with PDF cx^2 over $[0,1]$, zero elsewhere. What should c be?



Highlighted part $= 1$

$\int_0^1 cx^2\,dx = 1$

$\left(\frac{cx^3}{3}\right)\Big|_0^1 = 1$

$\frac{c}{3} - 0 = 1$

$\Rightarrow \boxed{c = 3}$

Is it OK that $f_X(x)$ is sometimes ≥ 1?

Here, when $x = 1$, $f_X(x) = 3$.

Yes. $f_X(x) \approx \mathbb{P}[X \in (x, x+dx)]$, drop dx

For $a \in [0,1]$, what is $\mathbb{P}[X \leq a]$? $\mathbb{P}[X < a]$?

$\mathbb{P}[X \leq a] = \int_{-\infty}^{a} f_X(x)\,dx$

$= \int_0^a 3x^2\,dx$

$= x^3\Big|_0^a = a^3$

$\mathbb{P}[X < a] = a^3$

because

$\mathbb{P}[X = a] = 0$

PDF

EX of an e

$0 \leq X \leq x$

$f_X(x)$

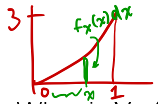
Not Too Different From Discrete I

Discrete RV:	**Continuous RV:**
$\mathbb{E}[X] = \sum_{a \in A} a \cdot \mathbb{P}[X = a]$	$\mathbb{E}[X] = \int_{-\infty}^{\infty} \underbrace{x}_{\text{"}a\text{"}} \cdot \underbrace{f_X(x)\,dx}_{\text{"}\mathbb{P}[X=a]\text{"}}$
$\mathbb{E}[X^2] = \sum_{a \in A} a^2 \cdot \mathbb{P}[X = a]$	$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} \underbrace{x^2}_{\text{"}a^2\text{"}} \cdot \underbrace{f_X(x)\,dx}_{\text{"}\mathbb{P}[X=a]\text{"}}$
$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$	$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

Fun With PDF II

Let X have PDF $3x^2$ over $[0,1]$, zero elsewhere.

What is $\mathbb{E}[X]$?



$\mathbb{E}[X] = \int_0^1 \underbrace{x}_{\text{"}a\text{"}}\underbrace{(3x^2\,dx)}_{\text{"}\mathbb{P}[X=a]\text{"}}$

$= \int_0^1 3x^3\,dx$

$= \left(\frac{3x^4}{4}\right)\Big|_0^1$

$= \frac{3}{4}$

What is $\text{Var}(X)$?

$\mathbb{E}[X^2] = \int_0^1 x^2 \cdot (3x^2\,dx)$

$= \int_0^1 3x^4\,dx$

$= \left(\frac{3x^5}{5}\right)\Big|_0^1 = \frac{3}{5}$

$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

$= \frac{3}{5} - \left(\frac{3}{4}\right)^2$

The CDF

The **cumulative distribution function** of X is:

$$F_X(\underset{\text{input}}{x}) = \mathbb{P}[X \leq \underset{\text{input}}{x}] = \int_{-\infty}^{x} f_X(y)\,dy$$

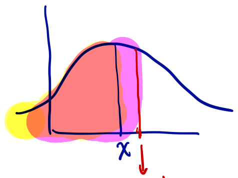
If I know the CDF, how do I recover the PDF?

$$f_X(x) = \frac{d}{dx}(F(x))$$

Sanity check:

1) $x \to \infty$
$F_X(x) = 1$

2) can the CDF be decreasing?
No

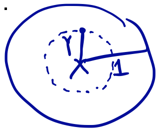


change in CDF
= pink
$\approx f_X(x)\,dx$

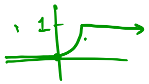
Fun With CDF: Dartboard

The CDF is often easier than the PDF!

I hit a random location on a circle, radius 1.
Let Y be the distance from the center.

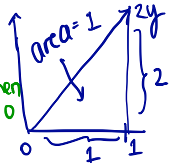


$$\mathbb{P}[Y \le y] = \frac{\text{area of radius } y \text{ circ}}{\text{area of whole circ.}} = \frac{\cancel{\pi} y^2}{\cancel{\pi}(1)^2} = y^2$$



$F_Y(y) =$ y^2 in $[0,1]$, 1 when $y \ge 1$, 0 when $y \le 0$

$f_Y(y) = \frac{d}{dy}(y^2) = 2y$



Break

What subjects should be taught in school but aren't?

Not Too Different From Discrete II

Discrete RV:
Tail Sum:
For X on non-neg. ints:

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \mathbb{P}[X \ge i]$$

Markov, Chebyshev Inequalities

Continuous RV:
Tail Sum:
For X on non-neg reals:

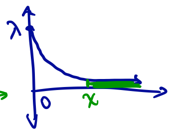
$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X \ge x]\, dx$$

Markov, Chebyshev Inequalities ↑ tomorrow

The Exponential RV

The continuous analogue of Geometric(p) RV.
We say X is an **exponential RV** if:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{else} \end{cases}$$



What is $\mathbb{P}[X \ge x]$ for $x \ge 0$?

↳ $\int_x^\infty \lambda e^{-\lambda x}\, dx$ → green area

$= (-e^{-\lambda x})\Big|_x^\infty = 0 - (-e^{-\lambda x}) = e^{-\lambda x}$

What is $F_X(x)$?

CDF → $F_X(x) = \mathbb{P}[X \le x]$

$= 1 - \mathbb{P}[X \ge x] = 1 - e^{-\lambda x}$

The Exponential RV

What is $\mathbb{E}[X]$? (Two Options)

1) use Def:
$\mathbb{E}[X] = \underbrace{\int_0^\infty x \cdot \lambda e^{-\lambda x}\, dx}_{\text{IBP}}$

2) use Tail Sum:
$\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X \ge x]\, dx$

$= \int_0^\infty e^{-\lambda x}\, dx$ ← prev. slide

$= \left(-\frac{1}{\lambda} e^{-\lambda x}\right)\Big|_0^\infty = 0 - \left(-\frac{1}{\lambda}\right)$

$= \frac{1}{\lambda}$

What is Var(X)? Hard to avoid IBP.
(In general, we avoid making you do IBP.)

Notes. Exercise.

$\frac{1}{\lambda^2}$

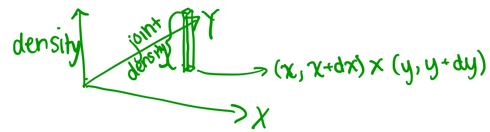
Not Too Different From Discrete III

Discrete RV:
Joint PMF:

$$\text{PMF}_{X,Y}(x,y) = \mathbb{P}[X = x, Y = y]$$

Continuous RV:
Joint PDF:

$$f_{X,Y}(x,y) \approx \mathbb{P}[X \in (x, x+dx), Y \in (y, y+dy)]$$



Joint Density

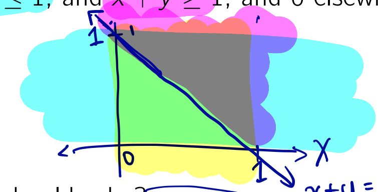
A func. $f : \mathbb{R}^2 \to \mathbb{R}$ is a **joint density** for X, Y if:

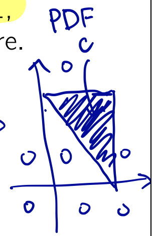
1. $f(x,y) \ge 0$ for all $x, y \in \mathbb{R}$.
   Discrete Analogue: $\mathbb{P}[X=a, Y=b] \ge 0$
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dx\, dy = 1$.
   Discrete Analogue: $\sum_{a \in A} \sum_{b \in B} \mathbb{P}[X=a, Y=b] = 1$

$\mathbb{P}[a \le X \le b, c \le Y \le d] =$ $\int_a^b \int_c^d f(x,y)\, dy\, dx$
Discrete Analogue:
$\sum_{i=a}^{b} \sum_{j=c}^{d} \mathbb{P}[X=i, Y=j]$

Fun With Joint Density!

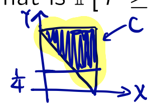
X, Y have joint density c when $0 \le x \le 1$, $0 \le y \le 1$, and $x + y \ge 1$, and 0 elsewhere.





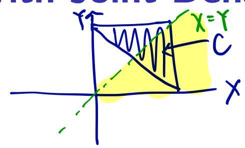
What should c be?
$1 = B \cdot h = (\frac{1}{2} \cdot 1 \cdot 1) \cdot c$ $\Rightarrow c = 2$

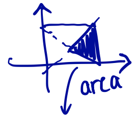
What is $\mathbb{P}[Y \ge \frac{1}{4}]$?



$\mathbb{P}[Y \ge \frac{1}{4}] = B \cdot h$ (little △)
$= (\frac{1}{2} - \frac{1}{32}) c = \frac{15}{16}$ (big △)

Fun With Joint Density!





What is $\mathbb{P}[X > Y]$? $\mathbb{P}[X > Y] = B \cdot h = \frac{1}{4} \cdot c$
$= \frac{1}{2}$

What is $\mathbb{P}[X \le x]$? $f_X(x)$?
Exercise.

Not Too Different From Discrete IV

We'll work with this more tomorrow...

Discrete RV:
X and Y are independent iff for all a, b:

$$\mathbb{P}[X = a, Y = b] = \mathbb{P}[X = a] \cdot \mathbb{P}[Y = b]$$

Continuous RV:
X and Y are independent iff for all $a \le b$, $c \le d$:

$$\mathbb{P}[a \le X \le b, c \le Y \le d] =$$

A Note on Independence

For continuous RVs, what is weird about the following?

$$\mathbb{P}[X = a, Y = b] = \mathbb{P}[X = a] \cdot \mathbb{P}[Y = b]$$

What we **can** do: consider a interval of length dx around a and b!

Summary

- Uses a **probability density**. Important events are **intervals** rather than particular values.
- Almost everything is **analogous to the discrete**! (Expectation, variance, inequalities, tail sum, joint distribution, independence)
- Sometimes, expectation and variance are easier to compute for continuous RVs!
- Tomorrow: more practice with **independence**, the exponential RV, conditioning...