Intro to Continuous RVs

CS 70, Summer 2019

Lecture 24, 8/5/19

So Far: The PMF

Every discrete distribution we've seen so far has a **probability mass function**.

For RV X:

$$\text{PMF}_X(a) = \mathbb{P}[X = a]$$

a can be any real number.
PMF_X is nonzero on...

Example:
Let $X \sim \text{Geometric}(p)$. Then, $\text{PMF}_X(a)$ is:

Uniform Over $[0, 1]$

What if X takes **uncountably** many values?

Example: I want X to be uniform over $[0, 1]$.
"Each $a \in [0, 1]$ equally likely."

What is $\mathbb{P}[X = 0]$?

What is $\mathbb{P}[X = 0.5]$?

What is $\mathbb{P}[a \leq X \leq b]$, for $0 \leq a \leq b \leq 1$?

A Discrete Approximation

Let's "approximate" X, a uniform RV over $[0, 1]$:

Choose integer m.
X_m take values:
Each value has probability:

$$\text{PMF}_X(a) =$$

A Discrete Approximation

What happens as we push $m \to \infty$?

$\mathbb{P}[a \leq X_m \leq b]$ gets closer and closer to

Width of each rectangle approaches 0.
In calculus, an **infinitesimally small width** is also known as:

If we attempt to write a PMF for X.

The PDF / "Density"

Drop the dx!

The **probability density function** (PDF) for X is a function $f : \mathbb{R} \to \mathbb{R}$ such that:

1. $f(x) \geq 0$ for all $x \in \mathbb{R}$.
   Discrete Analogue:
2. $\int_{-\infty}^{\infty} f(x)dx = 1$.
   Discrete Analogue:

$\mathbb{P}[a \leq X \leq b] =$
Discrete Analogue:

PMF vs. PDF: A Visual

Fun With PDF I

Let X be a RV with PDF cx^2 over $[0, 1]$, zero elsewhere. What should c be?

Is it OK that $f_X(x)$ is sometimes ≥ 1?

For $a \in [0, 1]$, what is $\mathbb{P}[X \leq a]$? $\mathbb{P}[X < a]$?

Not Too Different From Discrete I

Discrete RV:	**Continuous RV:**
$\mathbb{E}[X] = \sum_{a \in A} a \cdot \mathbb{P}[X = a]$	$\mathbb{E}[X] =$
$\mathbb{E}[X^2] = \sum_{a \in A} a^2 \cdot \mathbb{P}[X = a]$	$\mathbb{E}[X^2] =$
$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$	$\mathrm{Var}(X) =$

Fun With PDF II

Let X have PDF $3x^2$ over $[0, 1]$, zero elsewhere.

What is $\mathbb{E}[X]$?

What is $\mathrm{Var}(X)$?

The CDF

The **cumulative distribution function** of X is:

$$F_X(x) = \mathbb{P}[X \leq x] =$$

If I know the CDF, how do I recover the PDF?

$$f_X(x) =$$

Fun With CDF: Dartboard

The CDF is often easier than the PDF!

I hit a random location on a circle, radius 1.
Let Y be the distance from the center.

$$\mathbb{P}[Y \leq y] =$$

$$F_Y(y) =$$
$$f_Y(y) =$$

Break

What subjects should be taught in school but aren't?

Not Too Different From Discrete II

Discrete RV:
Tail Sum:
For X on non-neg. ints:

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \mathbb{P}[X \geq i]$$

Markov, Chebyshev Inequalities

Continuous RV:
Tail Sum:
For X on non-neg reals:

$$\mathbb{E}[X] =$$

Markov, Chebyshev Inequalities

The Exponential RV

The continuous analogue of Geometric(p) RV.
We say X is an **exponential RV** if:

$$f_X(x) =$$

What is $\mathbb{P}[X \geq x]$ for $x \geq 0$?

What is $F_X(x)$?

The Exponential RV

What is $\mathbb{E}[X]$? (Two Options)

What is $\text{Var}(X)$? Hard to avoid IBP.
(In general, we avoid making you do IBP.)

Not Too Different From Discrete III

Discrete RV:
Joint PMF:

$$\text{PMF}_{X,Y}(x, y) = \mathbb{P}[X = x, Y = y]$$

Continuous RV:
Joint PDF:

$$f_{X,Y}(x, y) = \mathbb{P}[X \in (x, x + dx), Y \in (y, y + dy)]$$

Joint Density

A func. $f : \mathbb{R}^2 \to \mathbb{R}$ is a **joint density** for X, Y if:

1. $f(x, y) \geq 0$ for all $x, y \in \mathbb{R}$.
   Discrete Analogue:
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx\, dy = 1$.
   Discrete Analogue:

$\mathbb{P}[a \leq X \leq b, c \leq Y \leq d] =$
Discrete Analogue:

Fun With Joint Density!

X, Y have joint density c when $0 \leq x \leq 1$, $0 \leq y \leq 1$, and $x + y \geq 1$, and 0 elsewhere.

What should c be?

What is $\mathbb{P}[Y \geq \frac{1}{4}]$?

Fun With Joint Density!

What is $\mathbb{P}[X > Y]$?

What is $\mathbb{P}[X \leq x]$? $f_X(x)$?

Not Too Different From Discrete IV

We'll work with this more tomorrow...

Discrete RV:
X and Y are independent iff for all a, b:

$$\mathbb{P}[X = a, Y = b] = \mathbb{P}[X = a] \cdot \mathbb{P}[Y = b]$$

Continuous RV:
X and Y are independent iff for all $a \leq b$, $c \leq d$:

$$\mathbb{P}[a \leq X \leq b, c \leq Y \leq d] =$$

A Note on Independence

For continuous RVs, what is weird about the following?

$$\mathbb{P}[X = a, Y = b] = \mathbb{P}[X = a] \cdot \mathbb{P}[Y = b]$$

What we **can** do: consider a interval of length dx around a and b!

Summary

- Uses a **probability density**. Important events are **intervals** rather than particular values.
- Almost everything is **analogous to the discrete**! (Expectation, variance, inequalities, tail sum, joint distribution, independence)
- Sometimes, expectation and variance are easier to compute for continuous RVs!
- Tomorrow: more practice with **independence**, the exponential RV, conditioning...