# Intro to Continuous RVs

CS 70, Summer 2019

Lecture 24, 8/5/19

# So Far: The PMF

Every discrete distribution we've seen so far has a **probability mass function**.

For RV $X$:

$$\text{PMF}_X(a) = \mathbb{P}[X = a]$$

$a$ can be any real number.
$\text{PMF}_X$ is nonzero on... all values that X takes

**Example:**
Let $X \sim \text{Geometric}(p)$. Then, $\text{PMF}_X(a)$ is:

$$\text{PMF}_X(a) = \begin{cases} 0 & \text{if } a \text{ is not positive integer} \\ (1-p)^{a-1}p & \text{else} \end{cases}$$

# Uniform Over $[0, 1]$

What if $X$ takes **uncountably** many values?

**Example:** I want $X$ to be uniform over $[0, 1]$.
"Each $a \in [0, 1]$ equally likely."

What is $\mathbb{P}[X = 0]$? 0.

one out of uncountably many #s.

What is $\mathbb{P}[X = 0.5]$? 0

What is $\mathbb{P}[\underbrace{a \leq X \leq b}_{b-a}]$, for $0 \leq a \leq b \leq 1$?


probability inside proportional to length

# A Discrete Approximation

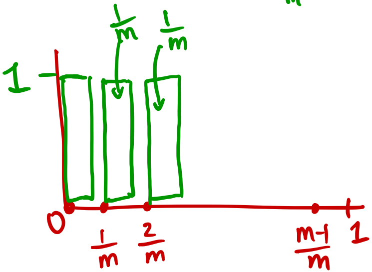
Let's "approximate" $X$, a uniform RV over $[0, 1]$:

Choose integer $m$. Take $m$ points "uniformly" in $[0,1]$

$X_m$ take values: $0, \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}$

Each value has probability: $\frac{1}{m}$

$$\mathrm{PMF}_{X_m}(a) = \begin{cases} 0 & \text{if } a \neq \frac{i}{m}, \text{ for } i = 0, 1, \ldots, m-1 \\ \frac{1}{m} & \text{else} \end{cases}$$




Area of rectangle = height × base

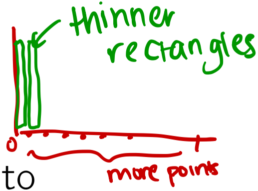
$\Downarrow$

probability mass

$= 1 \times \frac{1}{m}$

$= \frac{1}{m}$

# A Discrete Approximation




What happens as we push $m \to \infty$?

$\mathbb{P}[a \le X_m \le b]$ gets closer and closer to

$\hookrightarrow b-a$

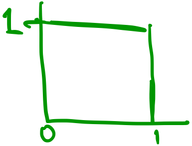
Width of each rectangle approaches 0.

In calculus, an **infinitesimally small width** is also known as: $dx$

(*) Similar to Riemann sum approx to integral.

If we attempt to write a PMF for $X$.

$$\text{PMF}_X(x) = \begin{cases} \underset{\substack{\downarrow \\ \text{height}}}{1} \cdot \underset{\substack{\downarrow \\ \text{width}}}{dx} & \text{for all } x \in [0,1] \\ 0 & \text{o.w.} \end{cases}$$



# The PDF / "Density"

Drop the $dx$!

The **probability density function** (PDF) for $X$ is a function $f : \mathbb{R} \to \mathbb{R}$ such that:

1. $f(x) \geq 0$ for all $x \in \mathbb{R}$.
   Discrete Analogue: all probabilities are non-neg.

2. $\int_{-\infty}^{\infty} f(x) dx = 1.$
   Discrete Analogue: all probabilities sum to 1.

height

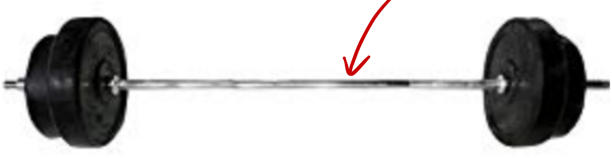
$\mathbb{P}[a \leq X \leq b] = \int_a^b f(x)\,dx$ width

Discrete Analogue: $\sum_{i=a}^{b} P[X=i]$

# PMF vs. PDF: A Visual

massless

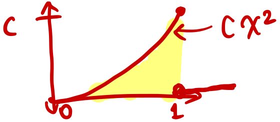
PMF



PDF

# Fun With PDF I

Let $X$ be a RV with PDF $cx^2$ over $[0,1]$, zero elsewhere. What should $c$ be?




Highlighted part = 1

$$\int_0^1 cx^2\,dx = 1$$

$$\left(\frac{cx^3}{3}\right)\Big|_0^1 = 1$$

$$\frac{c}{3} - 0 = 1$$

$$\Rightarrow \boxed{c = 3}$$

Is it OK that $f_X(x)$ is sometimes $\geq 1$?

Here, when $x = 1$, $f_X(x) = 3$.

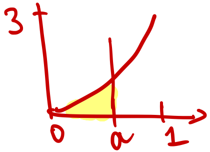
Yes. $f_X(x) \approx \mathbb{P}[X \in (x, x+dx)]$, drop $dx$

For $a \in [0,1]$, what is $\mathbb{P}[X \leq a]$? $\mathbb{P}[X < a]$?

$$\mathbb{P}[X \leq a] = \int_{-\infty}^{a} f_X(x)\,dx$$

$$= \int_0^a 3x^2\,dx$$

$$= x^3\Big|_0^a = a^3$$



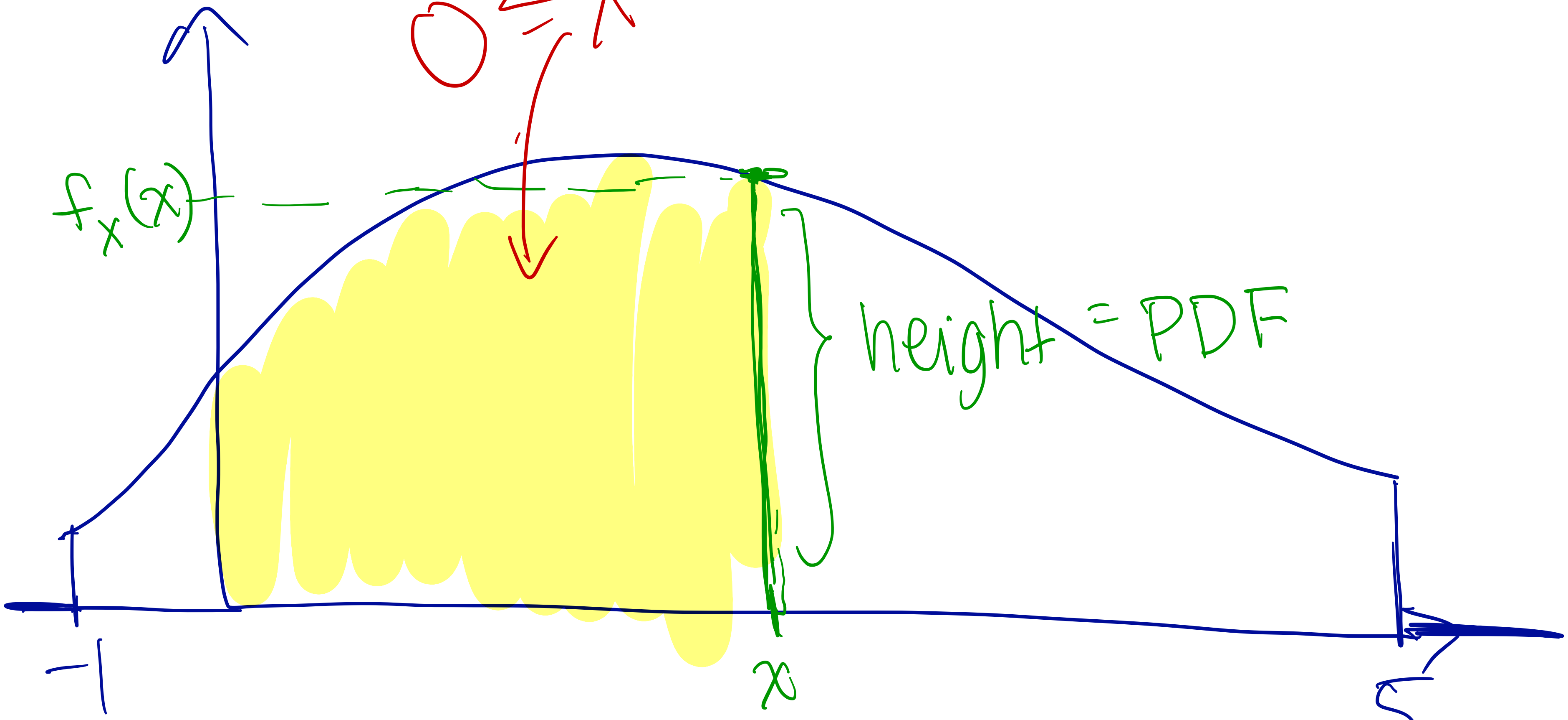

$\mathbb{P}[X < a] = a^3$

because

$\mathbb{P}[X = a] = 0$

PDF

Ex of an event: $0 \le X \le x$




$$\mathbb{P}[X = x] \approx \mathbb{P}[X \in [x, x+dx]]$$

$$\approx \underbrace{\text{height}}_{f_X(x)} \times \text{width} = f_X(x) \times dx$$

$\mathbb{P}[X = x] = 0$

# Not Too Different From Discrete I

**Discrete RV:**

$$\mathbb{E}[X] = \sum_{a \in A} a \cdot \mathbb{P}[X = a]$$

$$\mathbb{E}[X^2] = \sum_{a \in A} a^2 \cdot \mathbb{P}[X = a]$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

**Continuous RV:**

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} \underbrace{x}_{\text{"}a\text{"}} \cdot \underbrace{f_X(x)\,dx}_{\text{"}\mathbb{P}[X=a]\text{"}}$$

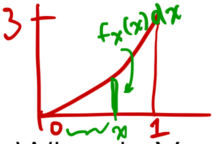
$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} \underbrace{x^2}_{\text{"}a^2\text{"}} \cdot \underbrace{f_X(x)\,dx}_{\text{"}\mathbb{P}[X=a]\text{"}}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

# Fun With PDF II

Let $X$ have PDF $3x^2$ over $[0, 1]$, zero elsewhere.

What is $\mathbb{E}[X]$?




$$\mathbb{E}[X] = \int_0^1 \overset{\text{"}a\text{"}}{x}\,(\underbrace{3x^2\,dx}_{\text{"}\mathbb{P}[X=a]\text{"}}) = \left(\frac{3x^4}{4}\right)\Big|_0^1$$

$$= \int_0^1 3x^3\,dx = \frac{3}{4}$$

What is $\mathrm{Var}(X)$?

$$\mathbb{E}[X^2] = \int_0^1 x^2 \cdot (3x^2\,dx)$$

$$= \int_0^1 3x^4\,dx$$

$$= \left(\frac{3x^5}{5}\right)\Big|_0^1 = \frac{3}{5}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \frac{3}{5} - \left(\frac{3}{4}\right)^2$$

# The CDF

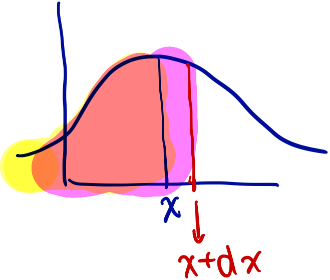

The **cumulative distribution function** of $X$ is:

input input

$$F_X(x) = \mathbb{P}[X \le x] = \int_{-\infty}^{x} f_X(y)\,dy$$

If I know the CDF, how do I recover the PDF?

$$f_X(x) = \frac{d}{dx}\left(F(x)\right)$$




change in CDF
= pink
$\approx f_X(x)\,dx$

sanity check:

1) $x \to \infty$
$F_X(x) = 1$

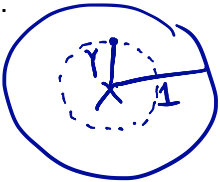
2) can the CDF be decreasing?
No

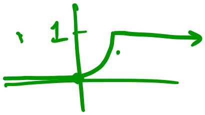
# Fun With CDF: Dartboard

The CDF is often easier than the PDF!

I hit a random location on a circle, radius 1.
Let $Y$ be the distance from the center.

$$\mathbb{P}[Y \le y] = \frac{\text{area of radius } y \text{ circ}}{\text{area of whole circ.}}$$

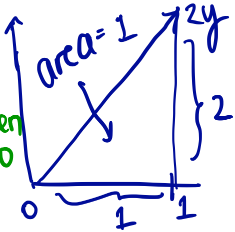



$$= \frac{\cancel{\pi} y^2}{\cancel{\pi}(1)^2} = y^2$$

$$F_Y(y) = y^2 \text{ in } [0,1], \quad 1 \text{ when } y \ge 1, \quad 0 \text{ when } y \le 0$$

$$f_Y(y) = \frac{d}{dy}(y^2) = 2y$$



# Break

What subjects should be taught in school but aren't?

# Not Too Different From Discrete II

**Discrete RV:**
Tail Sum:
For $X$ on non-neg. ints:

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \mathbb{P}[X \geq i]$$

Markov, Chebyshev Inequalities

**Continuous RV:**
Tail Sum:
For $X$ on non-neg reals:

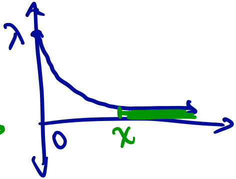
$$\mathbb{E}[X] = \int_0^{\infty} \mathbb{P}[X \geq x]\, dx$$

Markov, Chebyshev Inequalities

tomorrow

# The Exponential RV

The continuous analogue of Geometric($p$) RV.
We say $X$ is an **exponential RV** if:




$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{else} \end{cases}$$

What is $\mathbb{P}[X \geq x]$ for $x \geq 0$?

green area

$\hookrightarrow \int_x^\infty \lambda e^{-\lambda x}\, dx$

$= \left(-e^{-\lambda x}\right)\Big|_x^\infty = 0 - \left(-e^{-\lambda x}\right) = e^{-\lambda x}$

What is $F_X(x)$?

CDF

$F_x(x) = \mathbb{P}[X \leq x]$

$= 1 - \mathbb{P}[X \geq x] = 1 - e^{-\lambda x}$

# The Exponential RV

What is $\mathbb{E}[X]$? (Two Options)

1) Use Def:

$$\mathbb{E}[X] = \underbrace{\int_0^\infty x \cdot \lambda e^{-\lambda x} dx}_{\text{IBP}}$$

2) Use Tail Sum:

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X \geq x] dx$$

$$= \int_0^\infty e^{-\lambda x} dx \leftarrow \text{prev. slide}$$

$$= \left(-\frac{1}{\lambda} e^{-\lambda x}\right)\Big|_0^\infty = 0 - \left(-\frac{1}{\lambda}\right)$$

$$= \frac{1}{\lambda}$$

What is $\mathrm{Var}(X)$? Hard to avoid IBP.

(In general, we avoid making you do IBP.)

Notes. Exercise.

$$\frac{1}{\lambda^2}$$

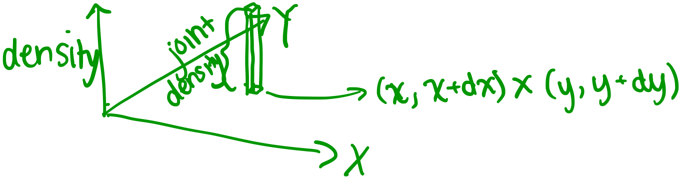
# Not Too Different From Discrete III

**Discrete RV:**
Joint PMF:

$$\mathrm{PMF}_{X,Y}(x,y) = \mathbb{P}[X = x, Y = y]$$

**Continuous RV:**
Joint PDF:

$$f_{X,Y}(x,y) \approx \mathbb{P}[X \in (x, x+dx), Y \in (y, y+dy)]$$



# Joint Density

A func. $f : \mathbb{R}^2 \to \mathbb{R}$ is a **joint density** for $X, Y$ if:

1. $f(x,y) \geq 0$ for all $x, y \in \mathbb{R}$.
   Discrete Analogue: $\mathbb{P}[X=a, Y=b] \geq 0$

2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dx\, dy = 1$.
   Discrete Analogue: $\sum_{a \in A} \sum_{b \in B} \mathbb{P}[X=a, Y=b] = 1$

$\mathbb{P}[a \leq X \leq b, c \leq Y \leq d] = \int_a^b \int_c^d f(x,y)\, dy\, dx$

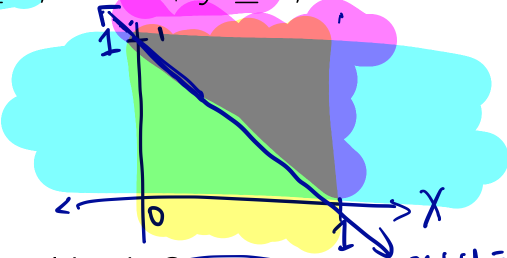
Discrete Analogue:

$$\sum_{i=a}^{b} \sum_{j=c}^{d} \mathbb{P}[X=i, Y=j]$$

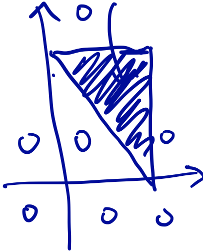
# Fun With Joint Density!

$X, Y$ have joint density $c$ when $0 \le x \le 1$, $0 \le y \le 1$, and $x + y \ge 1$, and 0 elsewhere.

PDF




$\Rightarrow$

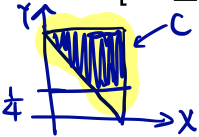



What should $c$ be?

$1 = Bh = \left(\frac{1}{2} \cdot 1 \cdot 1\right) \cdot c$

$\Rightarrow c = 2$

What is $\mathbb{P}[Y \ge \frac{1}{4}]$?



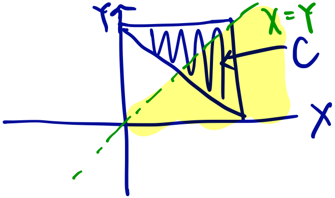

little $\Delta$

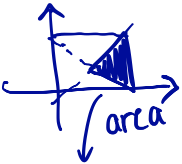
$\mathbb{P}[Y \ge \frac{1}{4}] = Bh$

$= \left(\frac{1}{2} - \frac{1}{32}\right)c = \frac{15}{16}$

big $\Delta$

# Fun With Joint Density!




What is $\mathbb{P}[X > Y]$?

$\mathbb{P}[X>Y] = B \cdot h = \frac{1}{4} \cdot C$
$= \frac{1}{2}$

What is $\mathbb{P}[X \leq x]$? $f_X(x)$?

Exercise.

# Not Too Different From Discrete IV

We'll work with this more tomorrow...

**Discrete RV:**
$X$ and $Y$ are independent iff for all $a, b$:

$$\mathbb{P}[X = a, Y = b] = \mathbb{P}[X = a] \cdot \mathbb{P}[Y = b]$$

**Continuous RV:**
$X$ and $Y$ are independent iff for all $a \leq b$, $c \leq d$:

$$\mathbb{P}[a \leq X \leq b, c \leq Y \leq d] =$$

# A Note on Independence

For continuous RVs, what is weird about the following?

$$\mathbb{P}[X = a, Y = b] = \mathbb{P}[X = a] \cdot \mathbb{P}[Y = b]$$

What we **can** do: consider a interval of length $dx$ around $a$ and $b$!

# Summary

- Uses a **probability density**. Important events are **intervals** rather than particular values.
- Almost everything is **analogous to the discrete**! (Expectation, variance, inequalities, tail sum, joint distribution, independence)
- Sometimes, expectation and variance are easier to compute for continuous RVs!
- Tomorrow: more practice with **independence**, the exponential RV, conditioning...