Intro to Markov Chains

CS 70, Summer 2019

Lecture 26, 8/7/19

Applications of Markov Chains

- Models systems of **states** and **transitions**
- PageRank – Google's search algorithm. States are webpages, transitions are links.
- Tons of applications outside of CS: statistical physics, speech recognition, bioinformatics, sabermetrics...

Markov Chain Definition

Three key components (and one assumption):

- Set $\mathcal{S}$ of
  Think of these as
- Transition probabilities.
  Think of these as
  Transitions **out of a node** should sum to
- Initial **distribution** $\mu^{(0)}$.
  Gives the probability that we start at a state.
- Memorylessness (aka **Markov property**)

Example: Gambling

I start with \$2. If I guess a coin flip correctly, I get \$1, and if I am incorrect, I lose \$1.
I stop gambling when I either hit \$0 or \$4.

Traversing the Chain

X_0 is the initial state.
Choose transitions according to its probability.

X_i is the state you're on at time i. X_i is a **RV**.

Markov Property:
Only the **current state** matters for the next.

"Knowing the entire history of the chain is equivalent to just knowing the current state."

Gambling II

Same chain as before:

What is $\mathbb{P}[X_1 = 3 | X_0 = 2]$?

What is $\mathbb{P}[X_{100} = 3 | X_{99} = 2, X_0 = 2]$?

What is $\mathbb{P}[X_1 = 3, X_2 = 2, X_3 = 3, X_4 = 4]$?

Gambling II

What is $\mathbb{P}[X_4 = 4]$?

The Transition Matrix

Calculations are easier to do when we stick the transition probabilities in a matrix.

Transition matrix P.
The (i, j) entry is $\mathbb{P}[X_1 = j | X_0 = i]$, or the **transition from** i **to** j.

The Distribution Vectors

So far: saw initial distribution $\mu^{(0)}$.
Can represent it as a **row vector:**

We can also define a **distribution at time** n:

Distribution at Time 1

We'll prove that $\mu^{(0)}P = \mu^{(1)}$.

If we know $\mu^{(1)}$, how do we get $\mu^{(2)}$?

Distribution at Time n

In general: $\mu^{(n)} = P^n \mu^{(0)}$. (Proof optional.)

Example: Two State Markov Chain

Aside: $n \to \infty$

For the two state Markov chain, as $n \to \infty$,

$$P^n \to$$

No matter what $\mu^{(0)}$ is:

Tomorrow: we'll study this in greater detail!

Break

Whats the weirdest thing youve ever eaten?

First Step Analysis: Two Heads

I repeatedly flip a coin, and stop when I get two heads in a row. What is the expected number of flips I need before stopping?

First Step Analysis: Two Heads

For state S, let $\tau(S)$ be the expected time to two heads, starting from state S.

Analyze a **single transition** out of each state to get the **first step equations**:

Max of Two Geometrics

Let $X, Y \sim \text{Geometric}(p)$. X, Y are **independent**. Say X, Y model time until a success. $\max(X, Y)$ is the first time that both X, Y have succeeded at least once. What is $\mathbb{E}[\max(X, Y)]$?

Max of Two Geometrics

Set up the first step equations, and solve:

Coupon Collector: A Markov Chain?

Can we reformulate Coupon Collector (with n distinct coupons) as a Markov chain?

How do we recover the expected number of coupons needed to get all n distinct ones?

Probability of $\mathcal{A}$ Before $\mathcal{B}$

Let $\mathcal{A}$ and $\mathcal{B}$ be two **disjoint** subsets of the states $\mathcal{S}$ of a Markov chain.

Let $\alpha(i)$ be the probability that we enter $\mathcal{A}$ before entering $\mathcal{B}$, if we start at state i.

Probability of $\mathcal{A}$ Before $\mathcal{B}$

Can also run first step analysis!

Gambling III

I start with \$100. In each round, I win \$100 with probability p and lose \$100 with probability $(1-p)$. I end when I either have \$0 or \$300.

What is the probability I end the game with \$300?

Gambling III

Let $\mathcal{A} = \{0\}, \mathcal{B} = \{300\}$. First step equations:

Summary

- Markov chains let you model real world problems with **states** and **transition probabilities**
- The **Markov property** tells you that where you go next only depends on the **current state**, not on any previous history.
- The first step analysis is a simple way of analyzing expected hitting times and probabilities of hitting certain states before others.