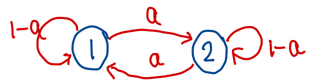
More Markov Chains: Classification of States, Stationary Distribution

CS 70, Summer 2019

Lecture 27, 8/8/19

The Symmetric Two-State Chain



$a \in (0,1)$

Transition matrix $P = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1-a & a \\ 2 & a & 1-a \end{array}$

Last time: $P^n = \begin{bmatrix} \underbrace{\frac{1}{2}+\frac{1}{2}(1-2a)^n}_{c} & \underbrace{\frac{1}{2}-\frac{1}{2}(1-2a)^n}_{c'} \\ c' & c \end{bmatrix}$ (notes)

As $n \to \infty$, $P^n \to \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Different Initial Distributions?

Let $\mu^{(0)} = [p \ \ (1-p)]$ be some initial distribution on the symmetric two state chain.

What is $\mu^{(n)} = \underbrace{\mu^{(0)} P^n}$ as $n \to \infty$?

$[p \ \ 1-p] \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = [\frac{1}{2} \ \ \frac{1}{2}]$

$\lim_{n\to\infty} \mu^{(n)}$

Observe: $\mu^{(0)} = [\frac{1}{2} \ \ \frac{1}{2}]$ is the only initial distribution such that $\mu^{(0)} = \lim_{n\to\infty} \mu^{(n)}$

Stationary Distribution

Let $\mathcal{S}, P$ be the states and transition matrix of a Markov chain. A distribution μ over states is **stationary** or **invariant** if

$\mu = \mu P$ ↳ rep. as vector [↑ ↑ ↑] indexes are states

Notation: π refers to a stationary dist.
$\pi_i \leftarrow i^{th}$ entry of π

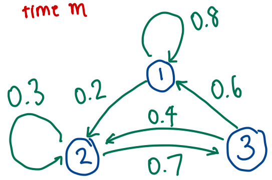
Intuition:

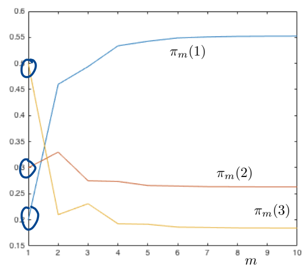
Always True: $\mu^{(n+1)} = \mu^{(n)} P$

when $n \to \infty$ $\approx \mu$ $\approx \mu$ $\Rightarrow \mu = \mu P$

Stationary Distribution: A Visual

$$\underset{\text{dist at time } m}{\pi_m} = \overset{\text{initial}}{\pi_0} P^m = \pi_0 \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.3 & 0.7 \\ 0.6 & 0.4 & 0 \end{bmatrix}^m \end{array}.$$



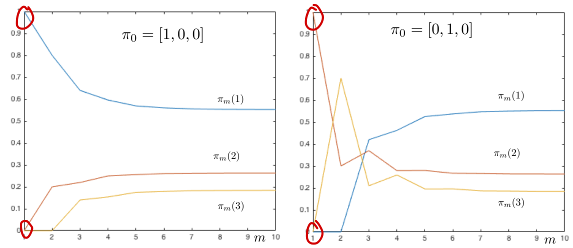


$\pi_0 = [0.2 \ \ 0.3 \ \ 0.5]$

$\pi \approx [0.55 \ \ 0.27 \ \ 0.18]$

Initial Distributions: A Visual

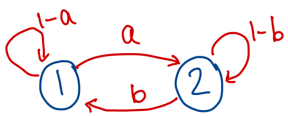
Same chain as prev. slide.



Start at state 1 Start at state 2.

Asymmetric Two State Chain

Similar example to the one before:



$a, b \in (0,1)$

$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$

Is there a stationary distribution? If so, what is it?

$\pi = [\pi_1 \;\; \pi_2]$: $\quad \pi P = \pi$

$$[\pi_1 \;\; \pi_2]\begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} = [\pi_1 \;\; \pi_2]$$

⇒ System of (linear) eqs:

① $\pi_1(1-a) + \pi_2 b = \pi_1$ → $\pi_2 b = \pi_1 a$

② $\pi_1 a + (1-b)\pi_2 = \pi_2$ → $\pi_1 a = \pi_2 b$

⊛ 3rd eq: $\pi_1 + \pi_2 = 1$

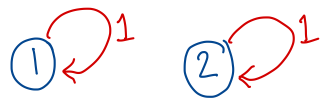
$\pi_1 = \frac{b}{a+b}$

$\pi_2 = \frac{a}{a+b}$

Loopy Two State Chain

$P = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{Identity Matrix}}$

A funny looking chain:



Is there a stationary distribution? If so, what is it?

$\pi = \pi P$

$= \pi I$ ← identity

$= \pi$

No matter what π is, it is stationary!

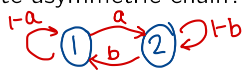
Q: When do we have a stationary distribution?
When do we have exactly 1?

Irreducibility

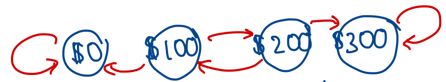
A Markov chain is **irreducible** we can go from every state $i \in \mathcal{S}$ to every other state $j \in \mathcal{S}$, possibly in **multiple steps**.

Are these chains **irreducible:**

Two state asymmetric chain? Yes $1 \to 2$ ✓ $2 \to 1$ ✓



Gambling chain (from yesterday)? No.



No way to go $0 \to$ anywhere!

Long Run Behavior

Let $\underbrace{I\{X_m = i\}}_{\text{event: at time } m\text{, I'm at state } i}$ be an indicator for whether $X_m = i$. (indicator)

How do we interpret the quantity below?

$$\frac{1}{n}\sum_{m=0}^{n-1} I\{X_m = i\} \to \text{fraction of time I spend in state } i$$

$$\sum_{m=0}^{n-1} \underset{\text{constant}}{I}\{X_m = i\} = \#\text{ times I'm at state } i \text{ from } t=0 \text{ to } n-1$$

What happens as $n \to \infty$?

Irreducibility Implies...

Theorem:
Let $\mathcal{S}, P$ be an **irreducible** Markov chain.
$\mathcal{S}$ is a **finite** set.

1. The stationary π exists and **is unique**. ($\pi = \pi P$)
2. For **any initial** $\mu^{(0)}$ and **all states** $i \in \mathcal{S}$:

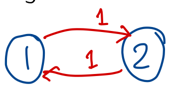
$$\lim_{n\to\infty} \underbrace{\frac{1}{n}\sum_{m=0}^{n-1} I\{X_m = i\}}_{\text{fraction of time in } i} = \underset{i^{\text{th}} \text{ entry of stationary}}{\pi_i}$$

Break

If you were a random variable, which one would you be and why?

Non-Loopy Two State Chain

A simple looking chain:



$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Is there a stationary distribution? If so, what is it?

$\pi = [\pi_1 \;\; \pi_2]$ $\qquad \pi P = \pi$ $\qquad \pi = [\tfrac{1}{2} \;\; \tfrac{1}{2}]$

$[\pi_1 \;\; \pi_2]\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [\pi_1, \pi_2]$

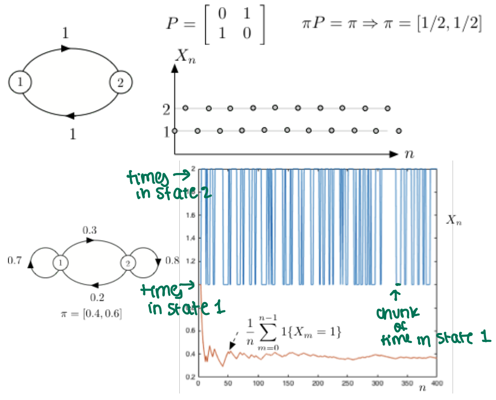
$\pi_1 \cdot 0 + \pi_2 \cdot 1 = \pi_1 \Rightarrow \pi_2 = \pi_1$

$\pi_1 + \pi_2 = 1$ $\Big\}$ $\pi_1 = \pi_2 = \tfrac{1}{2}$

If $X_0 = 1$, what is $\underbrace{X_{1000000}}_{1}$? What is $\underbrace{X_{1000001}}_{2}$?

even times $\qquad$ odd times.

Two Scenarios...

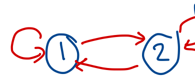


Periodicity

For a state i, its **periodicity** is the **gcd** of the length of all **tours** (i.e. walks from i to i).

Examples:

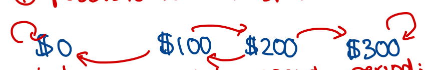
Asymmetric two state chain? walks from 1→1



lengths 1, 2, 3, ...
loop
gcd ⇒ period of state 1 is 1

Gambling chain from yesterday?

⊛ possible for diff states to have diff. periods!



\$0 period: 1 $\qquad$ \$100 period: 2 $\qquad$ \$200 period: 2 $\qquad$ \$300 period: 1

Periodicity + Irreducibility

Let $\mathcal{S}, P$ define an **irreducible** Markov chain.

1. Then, every state has the **same period**.

period = 1 for all states.

2. If the chain is also **aperiodic**, then as $n \to \infty$:

$$\mathbb{P}[X_n = i] \to \pi_i$$

lim n→∞

stronger than "fraction of states" statement.

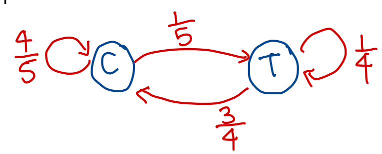
i^{th} entry of stationary dist.

Cars and Trucks

Three out of every four trucks on the road are followed by a car, while only one out of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

⊛ no explicit initial distrib.

Step 1: Draw the Markov Chain.



$$P = \begin{array}{c c} & \begin{array}{cc} C & T \end{array} \\ \begin{array}{c} C \\ T \end{array} & \begin{bmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} \end{array}$$

states: vehicles I see. X_t = vehicle at time t

Cars and Trucks

Step 2: Compute the stationary distribution.

$$P = \begin{array}{c c} & \begin{array}{cc} C & T \end{array} \\ \begin{array}{c} C \\ T \end{array} & \begin{bmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} \end{array}$$

$\pi = [\pi_C \;\; \pi_T]$

Because irreducible

π_C = frac. of vehicles that are cars

π_T = " " trucks

$[\pi_C \;\; \pi_T]\begin{bmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} = [\pi_C \;\; \pi_T]$ $\qquad$ ⊛ $\pi_C + \pi_T = 1$

$\frac{4}{5}\pi_C + \frac{3}{4}\pi_T = \pi_C \Rightarrow \frac{3}{4}\pi_T = \frac{1}{5}\pi_C$

$\Rightarrow \pi_C = \frac{15}{4}\pi_T$

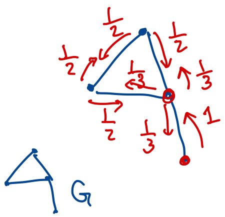
$\pi_T = \frac{4}{19}$

$\pi_C = \frac{15}{19}$

Markov Chain on a Graph

Let G be any loopless, connected graph. ← does NOT mean acyclic!
Each vertex represents a state, and at each vertex, we transition to a neighbor each with the same probability.

Q: Is this Markov chain irreducible?



Yes. G is connected.

For each edge, both directions present.

Markov Chain on a Graph

The unique stationary distribution π is given by:

$$\begin{array}{cc} & \begin{array}{cccc} v_1 & v_2 & v_3 & \end{array} \\ \text{index by vertices} \rightarrow \pi = & \left[\begin{array}{cccc} \frac{\deg(v_1)}{2|E|} & \frac{\deg(v_2)}{2|E|} & \frac{\deg(v_3)}{2|E|} & \dots \end{array} \right] \end{array}$$

$\sum_{v \in V} \deg(v) = 2|E|$

Can we verify this?

v_i column P

$$\left[\frac{\deg(v_1)}{2|E|} \quad \frac{\deg(v_2)}{2|E|} \quad \dots \right] \begin{array}{c} \\ \text{row } j \rightarrow \end{array} \left[\begin{array}{cc} \frac{1}{\deg(v_j)} & \text{if } (i,j) \in E \\ 0 & \text{o.w.} \end{array} \right] \overset{\text{Goal}}{=} \frac{\deg(v_i)}{2|E|}$$

$$\sum_{j \text{ nbrs of } i} \frac{\cancel{\deg(v_j)}}{2|E|} \cdot \frac{1}{\cancel{\deg(v_j)}} = \sum_{\text{nbrs of } i} \frac{1}{2|E|} = \frac{\deg(v_i)}{2|E|}$$

Sanity Check

Let G be a complete graph. K_n
What do we know about its long run behavior?

$$\pi = \left[\frac{1}{n} \quad \frac{1}{n} \quad \dots \quad \frac{1}{n} \right]$$

Aperiodic! Exercise

Let G be an odd cycle.
What do we know about its long run behavior?

Sanity Check

Let G be a hypercube.
What do we know about its long run behavior?

What fraction of time does it spend on strings with exactly k zeros?

Summary

- Stationary distributions do not change when we multiply them by the transition matrix.
- Irreducible chains always have a **unique** stationary distribution.
- We can say something about **fraction of time** spent in state i if a chain is irreducible
- If an irreducible chain is also aperiodic, the probability of being in a state at **any time** far enough out approaches π_i.

Next week: Conceptual review!