More Markov Chains: Classification of States, Stationary Distribution

CS 70, Summer 2019

Lecture 27, 8/8/19

The Symmetric Two-State Chain

Transition matrix $P =$

Last time: $P^n =$

As $n \to \infty$, $P^n \to$

Different Initial Distributions?

Let $\mu^{(0)} = [p \;\; (1-p)]$ be some initial distribution on the symmetric two state chain.

What is $\mu^{(n)} = \mu^{(0)} P^n$ as $n \to \infty$?

Observe: $\mu^{(0)} = [\frac{1}{2} \;\; \frac{1}{2}]$ is the only initial distribution such that

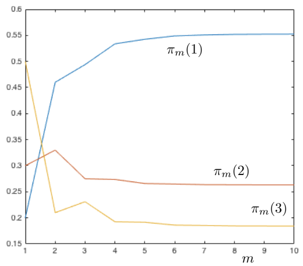
Stationary Distribution

Let $\mathcal{S}, P$ be the states and transition matrix of a Markov chain. A distribution μ over states is **stationary** or **invariant** if

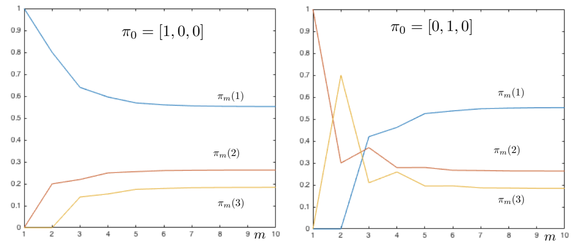
Intuition:

Stationary Distribution: A Visual

$$\pi_m = \pi_0 P^m = \pi_0 \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.3 & 0.7 \\ 0.6 & 0.4 & 0 \end{bmatrix}^m.$$



Initial Distributions: A Visual



Asymmetric Two State Chain

Similar example to the one before:

Is there a stationary distribution? If so, what is it?

Loopy Two State Chain

A funny looking chain:

Is there a stationary distribution? If so, what is it?

Q: When do we have a stationary distribution?
When do we have exactly 1?

Irreducibility

A Markov chain is **irreducible** we can go from every state $i \in \mathcal{S}$ to every other state $j \in \mathcal{S}$, possibly in **multiple steps**.

Are these chains **irreducible:**

Two state asymmetric chain?

Gambling chain (from yesterday)?

Long Run Behavior

Let $I\{X_m = i\}$ be an indicator for whether $X_m = i$.

How do we interpret the quantity below?

$$\frac{1}{n}\sum_{m=0}^{n-1} I\{X_m = i\}$$

What happens as $n \to \infty$?

Irreducibility Implies...

Theorem:
Let $\mathcal{S}, P$ be an **irreducible** Markov chain.
$\mathcal{S}$ is a **finite** set.
The stationary π exists and **is unique**.

For **any initial** $\mu^{(0)}$ and **all states** $i \in \mathcal{S}$:

Break

If you were a random variable, which one would you be and why?

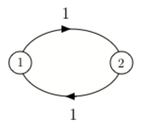
Non-Loopy Two State Chain

A simple looking chain:

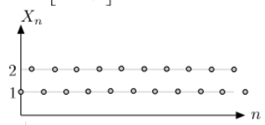
Is there a stationary distribution? If so, what is it?

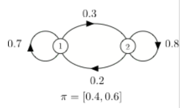
If $X_0 = a$, what is $X_{1000000}$? What is $X_{1000001}$?

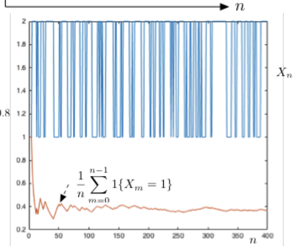
Two Scenarios...



$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \pi P = \pi \Rightarrow \pi = [1/2, 1/2]$







Periodicity

For a state i, its **periodicity** is the **gcd** of the length of all **tours** (i.e. walks from i to i).

Examples:
Asymmetric two state chain?

Gambling chain from yesterday?

Periodicity + Irreducibility

Let $\mathcal{S}, P$ define an **irreducible** Markov chain.
Then, every state has the **same period**.

If the chain is also **aperiodic**, then as $n \to \infty$:

$$\mathbb{P}[X_n = 1] \to \pi_i$$

Cars and Trucks

Three out of every four trucks on the road are followed by a car, while only one out of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

Step 1: Draw the Markov Chain.

Cars and Trucks

Step 2: Compute the stationary distribution.

Markov Chain on a Graph

Let G be any loopless, connected graph.
Each vertex represents a state, and at each vertex, we transition to a neighbor each with the same probability.

Q: Is this Markov chain irreducible?

Markov Chain on a Graph

The unique stationary distribution π is given by:

$$\pi =$$

Can we verify this?

Sanity Check

Let G be a complete graph.
What do we know about its long run behavior?

Let G be an odd cycle.
What do we know about its long run behavior?

Sanity Check

Let G be a hypercube.
What do we know about its long run behavior?

What fraction of time does it spend on strings with exactly k zeros?

Summary

- Stationary distributions do not change when we multiply them by the transition matrix.
- Irreducible chains always have a **unique** stationary distribution.
- We can say something about **fraction of time** spent in state i if a chain is irreducible
- If an irreducible chain is also aperiodic, the probability of being in a state at **any time** far enough out approaches π_i.

Next week: Conceptual review!