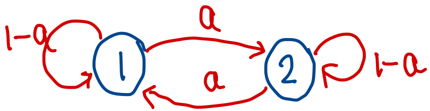
More Markov Chains: Classification of States, Stationary Distribution

CS 70, Summer 2019

Lecture 27, 8/8/19

The Symmetric Two-State Chain



$a \in (0,1)$

Transition matrix $P = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1-a & a \\ 2 & a & 1-a \end{array}$, i.e. $\begin{bmatrix} 1-a & a \\ a & 1-a \end{bmatrix}$

$C \qquad C'$

Last time: $P^n = \begin{bmatrix} \frac{1}{2}+\frac{1}{2}(1-2a)^n & \frac{1}{2}-\frac{1}{2}(1-2a)^n \\ C' & C \end{bmatrix}$

(notes)

As $n \to \infty$, $P^n \to \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Different Initial Distributions?

Let $\mu^{(0)} = [p \;\; (1-p)]$ be some initial distribution on the symmetric two state chain.

What is $\mu^{(n)} = \underbrace{\mu^{(0)} P^n}$ as $n \to \infty$?

$$[p \;\; 1-p] \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \left[\tfrac{1}{2} \;\; \tfrac{1}{2}\right]$$

$\lim_{n\to\infty} \mu^{(n)}$

Observe: $\mu^{(0)} = [\frac{1}{2} \;\; \frac{1}{2}]$ is the only initial distribution such that $\mu^{(0)} = \lim_{n\to\infty} \mu^{(n)}$

Stationary Distribution

Let $\mathcal{S}, P$ be the states and transition matrix of a Markov chain. A distribution μ over states is **stationary** or **invariant** if

rep. as vector

[↑ ↑ ↑]

indexes are states

$$\mu = \mu P$$

Notation: π refers to a stationary dist.

$\pi_i \leftarrow i^{th}$ entry of π

Intuition:

Always True: $\mu^{(n+1)} = \mu^{(n)} P$

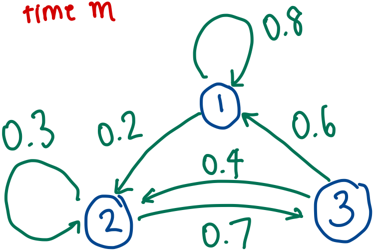
when $n \to \infty$: $\mu^{(n+1)} \approx \mu$, $\mu^{(n)} \approx \mu$ $\Rightarrow \mu = \mu P$

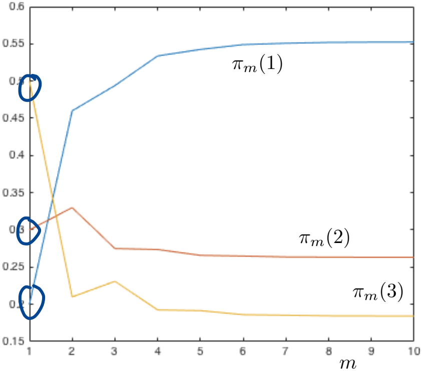
Stationary Distribution: A Visual

initial

$$\pi_m = \pi_0 P^m = \pi_0 \left(\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.8 & 0.2 & 0 \\ 2 & 0 & 0.3 & 0.7 \\ 3 & 0.6 & 0.4 & 0 \end{array} \right)^m .$$

dist at time m



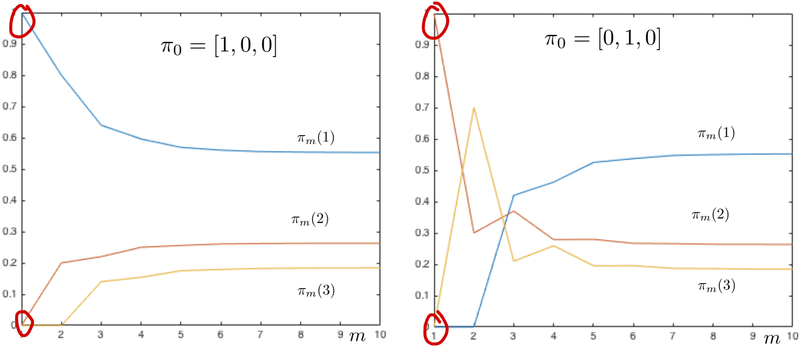


$\pi_0 = [0.2 \quad 0.3 \quad 0.5]$

$\pi \approx [0.55 \quad 0.27 \quad 0.18]$

Initial Distributions: A Visual

same chain as prev. slide.

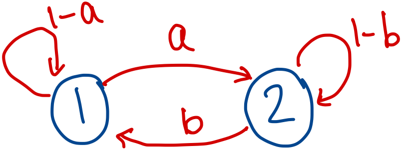


start at state 1

start at state 2.

Asymmetric Two State Chain

Similar example to the one before:



$a, b \in (0,1)$

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

Is there a stationary distribution? If so, what is it?

$\pi = [\pi_1 \quad \pi_2]$: $\qquad \pi P = \pi$

$$[\pi_1 \quad \pi_2] \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} = [\pi_1 \quad \pi_2]$$

$\Rightarrow$ System of (linear) eqs:

$$\left.\begin{aligned} &① \ \pi_1(1-a) + \pi_2 b = \pi_1 \\ &② \ \pi_1 a + (1-b)\pi_2 = \pi_2 \end{aligned}\right\} \begin{aligned} \pi_2 b &= \pi_1 a \\ \pi_1 a &= \pi_2 b \end{aligned}$$

⊛ 3rd eq: $\pi_1 + \pi_2 = 1$

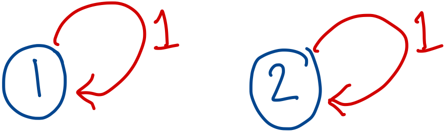
$$\pi_1 = \frac{b}{a+b}$$

$$\pi_2 = \frac{a}{a+b}$$

Loopy Two State Chain

$$P = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{Identity Matrix}}$$

A funny looking chain:



Is there a stationary distribution? If so, what is it?

$$\begin{aligned} \pi &= \pi P \\ &= \pi I \quad \text{identity} \\ &= \pi \end{aligned}$$

No matter what π is, it is stationary!

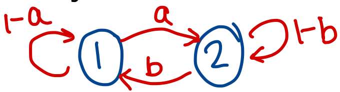
Q: When do we have a stationary distribution? When do we have exactly 1?

Irreducibility

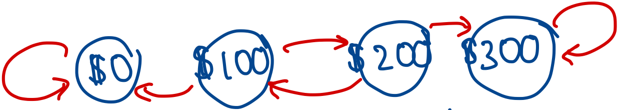
A Markov chain is **irreducible** we can go from every state $i \in \mathcal{S}$ to every other state $j \in \mathcal{S}$, possibly in **multiple steps**.

Are these chains **irreducible:**

Two state asymmetric chain? Yes $1 \to 2$ ✓ $2 \to 1$ ✓



Gambling chain (from yesterday)? NO.



No way to go $0 \to$ anywhere!

Long Run Behavior

indicator

Let $\underbrace{I\{X_m = i\}}_{\text{event: at time } m\text{, I'm at state } i}$ be an indicator for whether $X_m = i$.

How do we interpret the quantity below?

$$\frac{1}{n}\sum_{m=0}^{n-1} I\{X_m = i\} \rightarrow \text{fraction of time I spend in state } i$$

$\sum_{m=0}^{n-1} I\{X_m = i\}$ = # times I'm at state i from $t=0$ to $n-1$

constant

What happens as $n \to \infty$?

Irreducibility Implies...

Theorem:

Let $\mathcal{S}, P$ be an **irreducible** Markov chain.

$\mathcal{S}$ is a **finite** set.

1 The stationary π exists and **is unique**.

$\pi = \pi P$

2 For **any initial** $\mu^{(0)}$ and **all states** $i \in \mathcal{S}$:

$$\lim_{n \to \infty} \underbrace{\frac{1}{n} \sum_{m=0}^{n-1} I\{X_m = i\}}_{\text{fraction of time in } i} = \pi_i$$

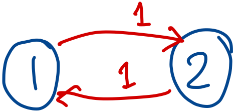
ith entry of stationary.

Break

If you were a random variable, which one would you be and why?

Non-Loopy Two State Chain

A simple looking chain:



$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Is there a stationary distribution? If so, what is it?

$\pi = [\pi_1 \ \ \pi_2]$ $\quad \pi P = \pi$ $\quad \pi = \left[\frac{1}{2} \ \ \frac{1}{2}\right]$

$[\pi_1 \ \pi_2]\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [\pi_1 \ \pi_2]$

$\pi_1 \cdot 0 + \pi_2 \cdot 1 = \pi_1 \Rightarrow \pi_2 = \pi_1$

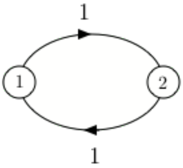
$\pi_1 + \pi_2 = 1$

$\pi_1 = \pi_2 = \frac{1}{2}$

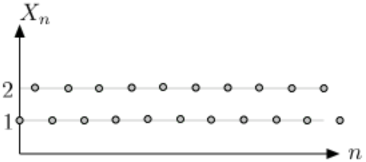
If $X_0 = \cancel{2}\,1$, what is $\underbrace{X_{1000000}}_{1}$? What is $\underbrace{X_{1000001}}_{2}$?

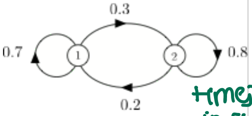
even times $\qquad$ odd times.

Two Scenarios...

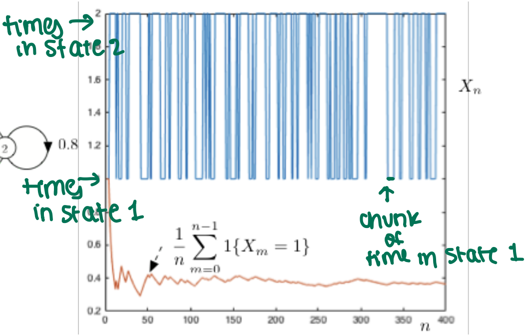


$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \pi P = \pi \Rightarrow \pi = [1/2, 1/2]$$





$\pi = [0.4, 0.6]$

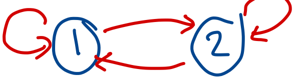


Periodicity

For a state i, its **periodicity** is the **gcd** of the length of all **tours** (i.e. walks from i to i).

Examples:

Asymmetric two state chain?



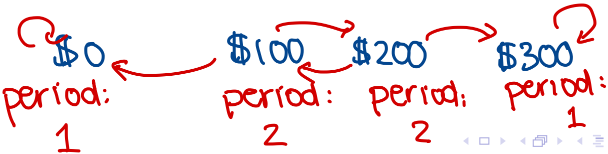
Walks from $1 \to 1$

lengths $\underbrace{1, 2, 3, \ldots}_{\text{gcd}}$ ⇒ period of state 1 is 1

↑ loop

Gambling chain from yesterday?

⊛ possible for diff states to have diff periods!



Periodicity + Irreducibility

Let $\mathcal{S}, P$ define an **irreducible** Markov chain.
1 Then, every state has the **same period**.

period = 1 for all states.

2 If the chain is also **aperiodic**, then as $n \to \infty$:

lim n→∞

$$\mathbb{P}[X_n = i] \to \pi_i$$

stronger than "fraction of states" statement.

ith entry of stationary dist.

Cars and Trucks

Three out of every four trucks on the road are followed by a car, while only one out of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

✱ no explicit initial distrib.

Step 1: Draw the Markov Chain.

$$\begin{tikzcd} C \arrow[loop left, "\frac{4}{5}"] \arrow[rr, bend left, "\frac{1}{5}"] & & T \arrow[loop right, "\frac{1}{4}"] \arrow[ll, bend left, "\frac{3}{4}"] \end{tikzcd}$$

$$P = \begin{array}{c|cc} & C & T \\ \hline C & \frac{4}{5} & \frac{1}{5} \\ T & \frac{3}{4} & \frac{1}{4} \end{array} \quad \left[\begin{matrix} \frac{4}{5} & \frac{1}{5} \\ \frac{3}{4} & \frac{1}{4} \end{matrix}\right]$$

states: vehicles I see. X_t = vehicle at time t

Cars and Trucks

Step 2: Compute the stationary distribution.

$$P = \begin{array}{c|cc} & C & T \\ \hline C & \frac{4}{5} & \frac{1}{5} \\ T & \frac{3}{4} & \frac{1}{4} \end{array}$$

$\pi = [\pi_C \quad \pi_T]$

Because irreducible

$\pi_C =$ frac. of vehicles that are cars

$\pi_T =$ " " trucks.

$$[\pi_c \ \pi_T]\begin{bmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} = [\pi_c \ \pi_T]$$

$(*)\ \pi_c + \pi_T = 1$

$$\frac{4}{5}\pi_c + \frac{3}{4}\pi_T = \pi_c \Rightarrow \frac{3}{4}\pi_T = \frac{1}{5}\pi_c$$

$$\Rightarrow \pi_c = \frac{15}{4}\pi_T$$

$\pi_T = \frac{4}{19}$

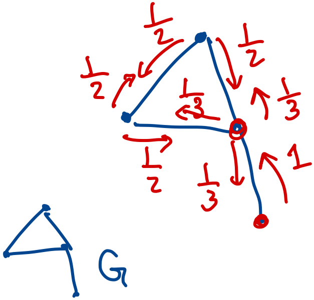
$\pi_C = \frac{15}{19}$

Markov Chain on a Graph

Let G be any loopless, connected graph. does NOT mean acyclic!

Each vertex represents a state, and at each vertex, we transition to a neighbor each with the same probability.

Q: Is this Markov chain irreducible?



Yes. G is connected.

For each edge, both directions present.

Markov Chain on a Graph

The unique stationary distribution π is given by:

index by vertices →

$$\pi = \begin{array}{cccc} \downarrow v_1 & \downarrow v_2 & \downarrow v_3 & \\ \left[\frac{\deg(v_1)}{2|E|} \right. & \frac{\deg(v_2)}{2|E|} & \frac{\deg(v_3)}{2|E|} & \left. \cdots \right] \end{array}$$

$\sum_{v \in V} \deg(v) = 2|E|$

Can we verify this?

$\swarrow v_i$ column P

$$\begin{bmatrix} \frac{\deg(v_1)}{2|E|} & \frac{\deg(v_2)}{2|E|} & \cdots \end{bmatrix} \begin{array}{c} \\ \text{row } j \rightarrow \\ \\ \end{array} \begin{bmatrix} \frac{1}{\deg(v_j)} \text{ if } (i,j) \in E \\ 0 \text{ o.w.} \\ \\ \end{bmatrix} \overset{\text{Goal}}{=} \frac{\deg(v_i)}{2|E|}$$

$$\sum_{j \text{ nbrs of } i} \frac{\cancel{\deg(v_j)}}{2|E|} \cdot \frac{1}{\cancel{\deg(v_j)}} = \sum_{\text{nbrs of } i} \frac{1}{2|E|} = \frac{\deg(v_i)}{2|E|}$$

Sanity Check

Let G be a complete graph. K_n
What do we know about its long run behavior?

$$\pi = \left[\frac{1}{n} \quad \frac{1}{n} \quad \cdots \quad \frac{1}{n}\right]$$

Aperiodic! Exercise

Let G be an odd cycle.
What do we know about its long run behavior?

Sanity Check

Let G be a hypercube.
What do we know about its long run behavior?

What fraction of time does it spend on strings with exactly k zeros?

Summary

- Stationary distributions do not change when we multiply them by the transition matrix.
- Irreducible chains always have a **unique** stationary distribution.
- We can say something about **fraction of time** spent in state i if a chain is irreducible
- If an irreducible chain is also aperiodic, the probability of being in a state at **any time** far enough out approaches π_i.

Next week: Conceptual review!