Lecture 28: Discrete Math Review

Or Is It Discreet Math?

Rough Outline

Today: review of first half of class

- Propositional Logic
- Proofs
- Graphs
- Modular Arithmetic
- Cryptography
- Polynomials
- Error Correcting Codes
- Countability
- Computability

Propositional Logic

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Allow simplification of complex statements

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- "Every integer is either even or odd."
- "$x + 3 = 7$."

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- "Pizza is a legitimate breakfast food." X
- "Every integer is either even or odd." ✓
- "$x + 3 = 7$." X

Make formulae w/operators: $\wedge, \vee, \neg, \implies$, etc
$(P \vee Q) \implies P$
$((\neg P) \iff Q) \wedge R$

Truth Tables

Formulae are really just functions!
Input: T/F values to propositions
Output: value of formula

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P	Q	$(\neg P) \vee (\neg Q)$	$\neg((\neg P) \vee (\neg Q))$	$P \wedge Q$
F	F			
F	T			
T	F			
T	T			

More on propositional / first order logic: Math 125A

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Proof by induction ($\forall n \in \mathbb{N}\ P(n)$):

- Prove $P(0)$ and $P(k) \implies P(k+1)$

Proof Poll

Do an example proof live! Poll for which one:

1. If $m|a$ and $n|b$, then $mn|ab$. (Direct)
2. Let $x \in \mathbb{Z}$. If $x^2 + 6x + 5$ is even, x is odd. (Contraposition)
3. Let r be rational and x be irrational. Then $r + x$ is irrational. (Contradiction)
4. Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then $(1 + x)^n \geq 1 + nx$. (Induction)

Direct Example

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Proof:

- Since $m|a$, $a = km$ for $k \in \mathbb{Z}$
- Since $n|b$, $b = jn$ for $j \in \mathbb{Z}$
- Hence $ab = km \cdot jn = kj(mn)$

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Proof:

- Contrapos: If x is even, $x^2 + 6x + 5$ is odd.
- Suppose $x = 2k$ for some $k \in \mathbb{Z}$
- $x^2+6x+5 = 4k^2+12k+5 = 2(2k^2+6k+2)+1$
- $2k^2 + 6k + 2 \in \mathbb{Z}$, so $x^2 + 6x + 5$ odd

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Proof:

- Suppose $r + x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$
- r rational, so $r = \frac{c}{d}$ for some $c, d \in \mathbb{Z}$
- Hence $x = (r + x) - r = \frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd}$
- So x rational, contradiction!

Induction Example

Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then $(1+x)^n \geq 1+nx$.

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Proof:

- Base Case: $n = 0$, statement is $1 \geq 1$.
- Suppose $(1+x)^k \geq 1 + kx$
- Then we have

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)^k(1+x) \\
&\geq (1+kx)(1+x) \\
&= 1 + x + kx + kx^2 \\
&= 1 + (k+1)x + kx^2 \\
&\geq 1 + (k+1)x
\end{aligned}$$

Graph Definitions

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Use drawings to help visualize

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Special kinds of graphs:

Complete

Bipartite

Hypercube

Planar

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Can induct on number of vertices, edges, etc

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"Shrink down, grow back" can help avoid this

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Example: proving Euler's formula $v + f = e + 2$

Euler and Coloring

Euler says planar graphs are sparse: $e \leq 6v - 12$
Means always have degree < 6 vertex!

Use to inductively prove 6-color theorem

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With more work, also gives 5-color theorem

Modular Arithmetic

Alternative to arithmetic on the real numbers

Define $+$ and $\cdot$ on $\{0, 1, 2, ..., m-1\}$

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More in depth look at this: Math 113, Math 115

Extended GCD Algorithm

Goal: find (d, a, b) st $\gcd(x, y) = d = ax + by$
Allows us to find inverses if $\gcd(x, y) = 1$!

Recursive call on $y, x \bmod y$ to get (d', a', b')
Return $(d', b', a' - \lfloor \frac{x}{y} \rfloor b')$

Chinese Remainder Theorem

Given coprime n_1, n_2, ..., n_k, $\exists$ unique soln modulo $N = \prod_i n_i$ to system of equations $x \equiv a_i \pmod{n_i}$

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Key is finding "basis" elements b_i st

- $b_i \equiv 1 \pmod{n_i}$
- $b_i \equiv 0 \pmod{n_j}$ for $j \neq i$

Private Key Cryptography

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One-Time Pad: xor message w/random, shared pad

Perfect security – but only for one message!

Public Key Cryptography

RSA: way to avoid logistical issues of OTP

Private key: $(N = pq, d)$
Public key: $(N, e = d^{-1} \pmod{(p-1)(q-1)})$

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Correctness: FLT + CRT
Security: ¯_(ツ)_/¯

Polynomial Representations

Two equiv representations of degree d polynomials:

- Coefficients $(c_d x^d + ... + c_1 x + c_0)$
- Values $((x_1, y_1), ..., (x_{d+1}, y_{d+1}))$

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Convert coefficients to values: evaluate polynomial

Other direction: Lagrange interpolation

Interpolation Interpretation

Given points (x_1, y_1), ..., (x_{d+1}, y_{d+1}), want degree d poly through them

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Given points (x_1, y_1), ..., (x_{d+1}, y_{d+1}), want degree d poly through them

Key is finding "basis" polys $\Delta_i(x)$ st

- $\Delta_i(x_i) = 1$
- $\Delta_i(x_j) = 0$ for $j \neq i$

Note similarity to proof of CRT!

Error Correcting Codes

Application of polys: fix transmission errors

Reed-Solomon: interpolate poly through message
$P(1) = m_1$, $P(2) = m_2$, ..., $P(n) = m_n$

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Recover P means recover message!
k erasures needs $n + k$ packets
k corruptions needs $n + 2k$ packets

Countability

Main idea: "same size" means "has bijection"

Use $\mathbb{N}$ as point of comparison
eg $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| = |\{0,1\}^*|$

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To prove a set countable:

- Provide bijection with known countable set
- Provide injection (1-1) to countable set
- Provide surjection (onto) from countable set

Last two from Cantor-Schröder-Bernstein Thm

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Not all sets are the same size as $\mathbb{N}$!

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Canonical Example: $\{0, 1\}^\infty$

n	$o(n)$
0	0 0 0 0 0 ...
1	1 0 1 0 1 ...
2	1 1 1 0 1 ...
3	0 1 0 0 0 ...
$\vdots$	$\vdots$

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$\vdots$	$\vdots$

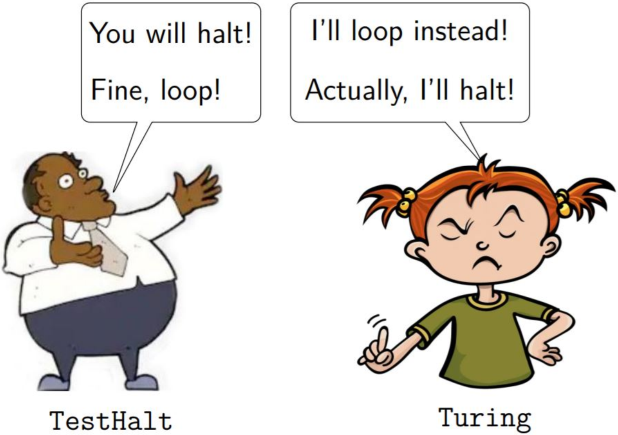
$s = 1101\ldots$

$s \neq o(n)$ for any n!

Show set uncountable w/diagonalization or show "same size"/"bigger than" known uncountable set

Uncomputability

Computers can't do everything!
Case study: Halting Problem is impossible



Reductions

Many other problems also uncomputable!

Often easiest to prove with reduction from TestHalt

Fin

Next time: probability review (with Elizabeth)!