Lecture 3: Induction

But then what is outduction?

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By original theorem,

$$\begin{aligned} |(x_1 + ... + x_{n-1}) + x_n| &\leq |x_1 + ... + x_{n-1}| + |x_n| \\ &\leq (|x_1| + ... + |x_{n-1}|) + |x_n| \end{aligned}$$

Induction Introduction

Principle of Induction: To prove $\forall n \in \mathbb{N}\ P(n)$, suffices to prove

(1) $P(0)$

(2) $\forall k \in \mathbb{N}\ [P(k) \implies P(k+1)]$

(1) is *base case* and (2) is *inductive step*.[1]

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- Suppose $|x_1 + \ldots + x_k| \leq |x_1| + \ldots + |x_k|$
- By the original triangle inequality, $|(x_1 + \ldots + x_k) + x_{k+1}| \leq |x_1 + \ldots + x_k| + |x_{k+1}|$

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- Combining these yields $|x_1 + \dots + x_{k+1}| \leq |x_1| + \dots + |x_{k+1}|$

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Theorem: For all $n \in \mathbb{N}$, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

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- Then $\sum_{i=0}^{k+1} i = \sum_{i=0}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1)$
- This equals $\frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2}$

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How many colors do we need to color a map (such that adjacent regions are different colors)?

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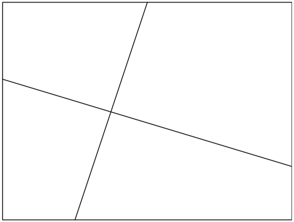

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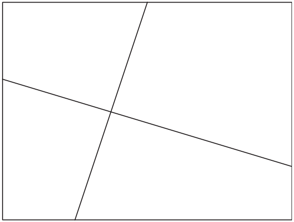

In this case, 2 colors will suffice!

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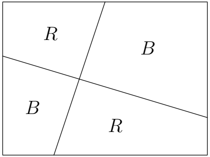
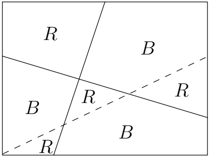
- Add line back, flip all colors on one side of it





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Knowing $P(k)$ isn't enough to get to $P(k+1)$!
Seem to be stuck :(

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Hmm, looks like the sum always works out to n^2...
Try proving it!

...and Prove It!

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Wait–this wasn't the theorem we wanted to prove! But new theorem implies old one.

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What we just did is called *strengthening the inductive hypothesis*.

General form: want to prove $\forall n\, P(n)$, instead prove $\forall n\, Q(n)$, where $Q(n) \implies P(n)$

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Look for patterns when strengthening.

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- Well drat...

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Stronger Theorem: $\forall n \in \mathbb{N}, \sum_{i=0}^{n} 2^{-i} = 2 - 2^{-n}$.

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- $\sum_{i=0}^{k+1} 2^{-i} = \sum_{i=0}^{k} 2^{-i} + 2^{-k-1} = 2 - 2^{-k} + 2^{-k-1}$
- $2^{-k} - 2^{-k-1} = 2^{-k-1}$

Break Time!

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Today's Discussion Question:
If you could eliminate one food so that no one would eat it ever again, what would you pick to destroy?

Other Fixes

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[4] Recall that this was an unproved lemma from last lecture.

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Strong Induction

Strong Inductive Principle: To prove $\forall n \in \mathbb{N}\ P(n)$, suffices to prove

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Same domino idea as regular induction — but now new domino pushed over by *all* previous ones

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Theorem: All $n \in \mathbb{N}$ st $n \geq 2$ have a prime factor.

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- $2 \leq a \leq k$, so a has a prime factor p
- Then p is a prime factor of $k + 1$

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Strong induction still useful–makes proofs easier!

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Recall recursion: function that calls itself

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Example: binary search

- Input: sorted list ℓ, target element e
- If len(ℓ) is 1, return true iff single element is e
- If center larger than e, recurse on left half
- If center smaller than e, recurse on right half
- If center is e, return true

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Issue: sets don't overlap when $k = 1$!

Fin

Next time: graph theory!