The Poisson Arrival Process

CS 70, Summer 2019

Bonus Lecture, 8/14/19

Poisson Distribution: Review

Values: non-neg integers

Parameter(s): λ, "rate"

$\mathbb{P}[X = i] = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$

$\mathbb{E}[X] = \lambda$

$\mathrm{Var}[X] = \lambda$

Poisson Over Time

Let $B_1 \sim \text{Poisson}(\lambda)$ be the number of bikes that are stolen on campus in one hour. (Go bears?)

What is the distribution of $B_{2.5}$, the number of bikes that are stolen on campus in two hours? and a half

$B_{2.5} \sim \text{Poisson}(2.5\lambda)$

$\mathbb{E}[B_{2.5}] = 2.5\lambda$

Rate over time $T =$ $T \cdot \lambda$

Adding Poissons: Review

Let $T_1 \sim \text{Poisson}(\lambda_1)$ be the number of particles detected by Machine 1 over 3 hours.

Let $T_2 \sim \text{Poisson}(\lambda_2)$ be the number of particles detected by Machine 2 over 4 hours.

The machines run **independently**.

What is the distribution of $T_1 + T_2$?

$$T_1 + T_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$$

Adding Poissons: Twist?

What is the distribution of the **total number of particles detected across both machines** over 5 hours?

$T_1' = \#$ particles from M1 in 1 hour
$T_2' =$ " " " M2 " "

$T_1' \sim \text{Poisson}\left(\frac{\lambda_1}{3}\right)$
$T_2' \sim \text{Poisson}\left(\frac{\lambda_2}{4}\right)$

1 hour: $T_1' + T_2' \sim \text{Poi}\left(\frac{\lambda_1}{3} + \frac{\lambda_2}{4}\right)$

$\downarrow$

5 hour: $\sim \text{Poi}\left[5\left(\frac{\lambda_1}{3} + \frac{\lambda_2}{4}\right)\right]$

Decomposing Poissons

Let $T \sim \text{Poisson}(\lambda)$ be the number of particles detected by a machine over one hour.

Each particle behaves **independently** of others.

Each detected particle is an α-particle with probability p, and a β-particle otherwise.

Let T_α be the number of α-particles detected by a machine over one hour. What is its distribution?

Decomposing Poissons

Goal: $\mathbb{P}[T_\alpha = a]$

Let T_α be the number of α-particles detected by a machine over one hour. What is its distribution?

$$\mathbb{P}[T_\alpha = a] = \sum_{n=a}^{\infty} \mathbb{P}[(T_\alpha = a)\cap(T=n)] \quad \leftarrow \text{Total Prob.}$$

$$= \sum_{n=a}^{\infty} \underbrace{\left(e^{-\lambda}\cdot\frac{\lambda^n}{n!}\right)}_{T=n} \binom{n}{a} p^a (1-p)^{n-a}$$

placement of α's

$$= \sum_{n=a}^{\infty} e^{-\lambda p} e^{-\lambda(1-p)} \cdot \frac{\lambda^a \cdot \lambda^{n-a}}{\cancel{n!}} \cdot \left(\frac{\cancel{n!}}{a!(n-a)!}\right) p^a (1-p)^{n-a}$$

$$= \left(e^{-\lambda p}\cdot\frac{(\lambda p)^a}{a!}\right)\left(e^{-\lambda(1-p)}\right)\underbrace{\sum_{n=a}^{\infty}\frac{[\lambda(1-p)]^{n-a}}{(n-a)!}}_{\text{Taylor series for } e^{\lambda(1-p)}}$$

$$= e^{-\lambda p}\cdot\frac{(\lambda p)^a}{a!}$$

How about T_β, the number of β-particles?

$T_\alpha \sim \text{Poisson}(\lambda p)$ $\qquad$ $T_\beta \sim \text{Poisson}(\lambda(1-p))$

Independence?

Are T_α and T_β independent? Yes.

Decomposing Poissons Remix

Now there are 3 kinds of particles: α, β, γ.

Each detected particle behaves independently of others, and is α with probability p, β with probability q, and γ otherwise.

$$T_\alpha \sim \text{Poisson}(\lambda p)$$

$$T_\beta \sim \text{Poisson}(\lambda q)$$

$$T_\gamma \sim \text{Poisson}(\lambda(1-p-q))$$

Punt: T_α, T_β, T_γ are **mutually independent**.
Sanity Check: $T_\alpha + T_\beta + T_\gamma \sim \text{Poisson}(\lambda)$

Exponential Distribution: Review

Values: $[0, \infty)$

Parameter(s): λ

"$\cancel{\mathbb{P}[X = i]}$" $=$ PDF: $f_X(x) = \lambda e^{-\lambda x}$

$\mathbb{E}[X] = \frac{1}{\lambda}$

$\mathrm{Var}[X] = \frac{1}{\lambda^2}$

Break

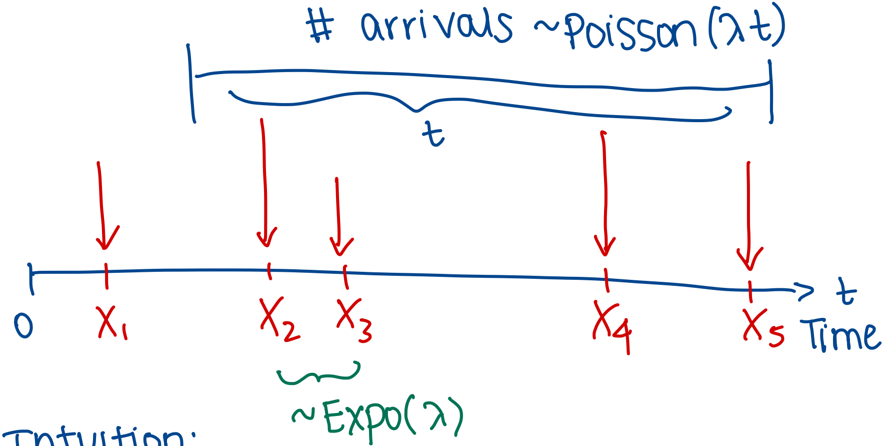
If you could rename the Poisson RV (or any RV for that matter), what would you call it?

Poisson Arrival Process Properties

We'll now work with a specific setup:

- There are **independent** "arrivals" over time.
- The time between consecutive arrivals is Expo(λ). We call λ the **rate**.
  Times between arrivals also **independent**.
- For a time period of length t, the **number of arrivals** in that period is Poisson(λt).
- Disjoint time intervals have independent numbers of arrivals.

Poisson Arrival Process: A Visual



Intuition:

$\mathbb{E}[\text{inter-arrival time}] = \frac{1}{\lambda}$

Unit time ⇒ see $\dfrac{\overset{\text{unit time}}{1}}{\underset{\text{inter-arrival}}{\frac{1}{\lambda}}} = \lambda = \mathbb{E}[\text{Poisson}(\lambda)]$

Transmitters I

A transmitter sends messages according to a Poisson Process with hourly rate λ.

Given that I've seen 0 messages at time t, what is the expected time until I see the first?

$$X_1 \sim \text{Expo}(\lambda)$$

Memorylessness: $\mathbb{P}[X \geq s+t \mid X \geq t] = \mathbb{P}[X \geq s]$

At time t, can "reset"

Treat time t as time 0.

Expected first arrival $= \frac{1}{\lambda}$ time <u>after</u> t

Transmitters I

How many messages should I expect to see from 12:00-2:00 and 5:00-5:30?

Total Time: 2h + 0.5h

= 2.5h

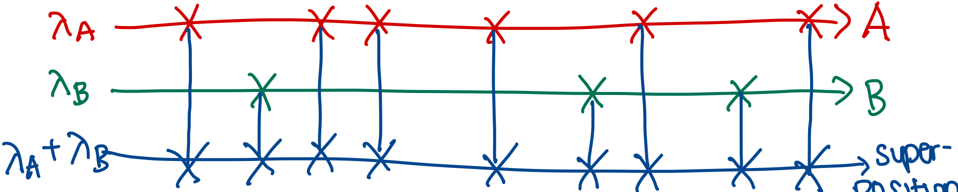
arrivals in both intervals ~ Poisson(2.5λ)

$\Rightarrow E[\text{\# messages}] = 2.5\lambda$

Transmitters II: Superposition

Transmitters A, B sends messages according to Poisson Processes of rates λ_A, λ_B respectively. The two transmitters are **independent**.

We receive messages from both A and B.



What is the expected amount of time until the first message from either transmitter?

$T \sim \text{Expo}(\lambda_A + \lambda_B)$ $\quad E[T] = \frac{1}{\lambda_A + \lambda_B}$

Transmitters II: Superposition

Transmitters A, B sends messages according to Poisson Processes of rates λ_A, λ_B respectively. The two transmitters are **independent**.

We receive messages from both A and B.

repeat

What is the expected amount of time until the first message from either transmitter?

Transmitters II: Superposition

If the messages from A all have 3 words, and the messages from B all have 2 words, how many words do we expect to see from 12:00-2:00?

M_A = # messages from A, 12:00-2:00
M_B = " " " B, "

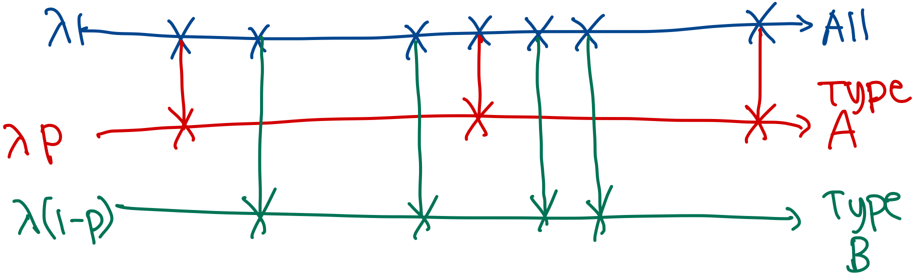
$M_A \sim \text{Poisson}(\lambda_A \cdot 2)$
$M_B \sim \text{Poisson}(\lambda_B \cdot 2)$

$$\mathbb{E}[\text{words}] = \mathbb{E}[3M_A + 2M_B]$$
$$= 3\mathbb{E}[M_A] + 2\mathbb{E}[M_B] = 6\lambda_A + 4\lambda_B$$

Kidney Donation: Decomposition

My probability instructor's favorite example...

Kidney donations at a hospital follow a Poisson Process of rate λ per day. Each kidney either comes from blood type A or blood type B, with probabilities p and $(1-p)$ respectively.



Kidney Donation: Decomposition

If I have blood type B, how long do I need to wait before receiving a compatible kidney?

Type B: Poisson Process rate $\lambda(1-p)$

T = time until first B. $T \sim \text{Expo}(\lambda(1-p))$ $\mathbb{E}[T] = \frac{1}{\lambda(1-p)}$

Say I just received a type A kidney.

The patient receiving a type A kidney after me is expected to live 50 more days without a kidney donation. What is the probability they survive?

T = time until next A kidney.

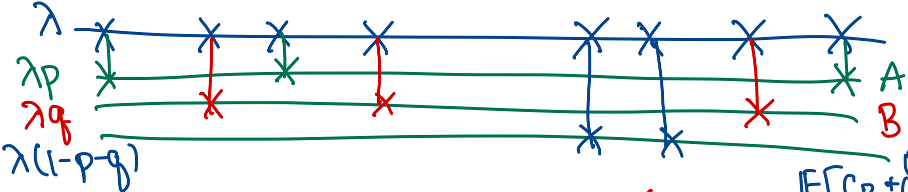
$T \sim \text{Expo}(\lambda p)$

$$\mathbb{P}[T \le 50] = \int_0^{50} \lambda p e^{-\lambda p x} dx = 1 - e^{-(\lambda p)(50)}$$

Kidney Donation: Decomposition

Now imagine kidneys are types A, B, O with probabilities p, q, $(1-p-q)$, respectively.

If I have type B blood, I can receive both B and O. How many compatible kidneys do I expect to see over the next 3 days?



C_B = # B kidneys, 3 days $\sim \mathrm{Poi}(3\lambda q)$

C_O = # O " " " $\sim \mathrm{Poi}(3\lambda(1-p-q))$

$\mathbb{E}[C_B + C_O] = 3\lambda(1-p)$

Summary

When working with **time**, use Expo(λ) RVs.

When working with **counts**, use Poisson(λ) RVs.

Superposition: combine independent Poisson Processes, **add** their rates.

Decomposition: break Poisson Process with rate λ down into rates $p_1\lambda$, $p_2\lambda$, and so on, where p_i's are probabilities.