Internship

Lecture 31: Stable ~~Marriage~~ Algorithm

Heteronormativity Is Dumb.

Announcements

End of course survey should now have all staff members!
Due by 11:59 PM tomorrow — pls fill it out!

Problem Statement

4 students applying for internships
4 companies want 1 intern each

Everyone has a preference:

Stdnt	Preferences
A	$4 > 3 > 1 > 2$
B	$3 > 4 > 1 > 2$
C	$4 > 1 > 3 > 2$
D	$3 > 2 > 1 > 4$

Comp	Preferences
1	$B > D > C > A$
2	$B > D > A > C$
3	$B > A > D > C$
4	$B > C > A > D$

Who should work where?

Bad Matching

Stdnt	Preferences
A	$4 > 3 > 1 > 2$
B	$3 > 4 > 1 > 2$
C	$4 > 1 > 3 > 2$
D	$3 > 2 > 1 > 4$

Comp	Preferences
1	$B > D > C > A$
2	$B > D > A > C$
3	$B > A > D > C$
4	$B > C > A > D$

Should B work at 1?

B wants to work at 3, 3 wants B
Incentive for both to leave system

Want to avoid this kind of problem

Stability

Rogue pair is company + student that prefer each other over assigned counterpart

Matching **stable** if no rogue pairs

Goal: Given preference lists, find stable pairing

Is It Stable?

Stdnt	Preferences
A	$4 > 3 > 2 > 1$
B	$3 > 4 > 1 > 2$
C	$4 > 1 > 3 > 2$
D	$3 > 2 > 1 > 4$

Comp	Preferences
1	$B > D > C > A$
2	$B > D > A > C$
3	$B > A > D > C$
4	$B > C > A > D$

Is $(A, 4), (B, 3), (C, 1), (D, 2)$ stable?
No — $(4, C)$ is rogue!

What about $(A, 1), (B, 3), (C, 4), (D, 2)$?
Yep!

Is Stability Guaranteed?

Natural Q: is there always a stable matching?
Not immediately obvious!

Consider "Stable Roomates":

Person	Preferences
A	$B > C > D$
B	$C > A > D$
C	$A > B > D$
D	$A > B > C$

Possible pairings:

- $(A, B), (C, D)$
- $(A, C), (B, D)$
- $(A, D), (B, C)$

Gale-Shapley Algorithm

Turns out, internships always has stable matching!
Prove by giving algorithm to find one

Morning: Students apply to top company on list
Afternoon: Companies reject all but top applicant
Evening: Rejected students cross off company

Algorithm stops once no rejections.

Claim: Algorithm always terminates
No more than n^2 rejections possible!

Example Run Day 1

Stdnt	Preferences	Comp	Preferences
A	$4 > 3 > 2 > 1$	1	$B > D > C > A$
B	$3 > 4 > 1 > 2$	2	$B > D > A > C$
C	$4 > 1 > 3 > 2$	3	$B > A > D > C$
D	$3 > 2 > 1 > 4$	4	$B > C > A > D$

Example Run Day 2

Stdnt	Preferences	Comp	Preferences
A	~~4~~ $> 3 > 2 > 1$	1	$B > D > C > A$
B	$3 > 4 > 1 > 2$	2	$B > D > A > C$
C	$4 > 1 > 3 > 2$	3	$B > A > D > C$
D	~~3~~ $> 2 > 1 > 4$	4	$B > C > A > D$

Example Run Day 3

Stdnt	Preferences	Comp	Preferences
A	~~4~~ $>$ ~~3~~ $> 2 > 1$	1	$B > D > C > A$
B	$3 > 4 > 1 > 2$	2	$B > D > A > C$
C	$4 > 1 > 3 > 2$	3	$B > A > D > C$
D	~~3~~ $> 2 > 1 > 4$	4	$B > C > A > D$

Example Run Day 3

Stdnt	Preferences	Comp	Preferences
A	~~4~~ $>$ ~~3~~ $>$ ~~2~~ > 1	1	$B > D > C > A$
B	$3 > 4 > 1 > 2$	2	$B > D > A > C$
C	$4 > 1 > 3 > 2$	3	$B > A > D > C$
D	~~3~~ $> 2 > 1 > 4$	4	$B > C > A > D$

Algorithm terminates with matching
$(A, 1), (B, 3), (C, 4), (D, 2)$

Stable here — how do we know it always is?

Improvement Lemma

Say company "interviewing" student if student applies and not yet rejected

Lemma: If S applies to C on day k, C interviewing S or better on every subsequent day

Proof:

- Base Case: S applies on day k, so best applicant S or better
- Suppose interviews S' $\geq$ S on day $j \geq k$
- S' applies on day $j+1$, so best $\geq$ S' $\geq$ S

Lemma Not Cool Enough To Have Name

Lemma: Applications on last day form a pairing

Proof:

- No rejections, so ≤ 1 applicant per job
- Only poss issue if student rejected everywhere!
- That student applied everywhere
- Improvement Lemma: comps have better stdnt
- Would need more students than companies!

Now just have to prove no rogue couples!

Wrapping It Up

Theorem: Matching at end of algo is stable

Proof:

- Suppose have (S, C) matched, (S, C*) rogue
- Def of rogue: S likes C* > C
- So in algorithm S applied to C*
- Improvement Lemma: C* has better than S
- So C* wouldn't go rogue — contradiction!

Our Final Break :'(

Time to take a break!

Two options:

- Normal discussion question
- I can show you a magic trick

Today's Discussion Question:
Is a hot dog a taco?

Magic Trick

Optimality

Is the stable pairing we get good? What is "good"?

Def: Optimal company for S is best they can get *in any stable pairing*

Not necessarily top of their list!

Stdnt	Preferences
A	$4 > 3 > 2 > 1$
B	$3 > 4 > 1 > 2$
C	$4 > 1 > 3 > 2$
D	$3 > 2 > 1 > 4$

Comp	Preferences
1	$B > D > C > A$
2	$B > D > A > C$
3	$B > A > D > C$
4	$B > C > A > D$

Optimal Pairing

Theorem: Pairing from algorithm gives all students their optimal company

Proof:
Show: no student rejected by opt company on day k

- Base Case: Day 1
- Suppose S rejected by C* in favor of S'
- C* opt for S, so have stable pairing w/(S, C*)
- S' has company C' in that pairing
- S' applies to C* on first day, so C* $\geq$ C'
- C* rejects S, so S' $\geq$ S
- (S', C*) rogue — contradiction!

Optimality Inductive Step

Just need (strong) inductive step:
If no student rejected by opt company day k or earlier, none on day $k+1$

Proof:

- Suppose S rejected by C* in favor of S'
- C* opt for S, so have stable pairing w/(S, C*)
- S' has company C' in pairing; opt company C'*
- Ind Hypothesis: S' not rejected by C'*
- So for S', C* $\geq$ C'* $\geq$ C'
- C* rejects S, so S' $\geq$ S
- (S', C*) rogue — contradiction!

Pessimality

What is opposite of optimal?

Def: Pessimal student for C is worst they get *in any stable pairing*

Not necessarily bottom of their list!

Stdnt	Preferences
A	$4 > 3 > 2 > 1$
B	$3 > 4 > 1 > 2$
C	$4 > 1 > 3 > 2$
D	$3 > 2 > 1 > 4$

Comp	Preferences
1	$B > D > C > A$
2	$B > D > A > C$
3	$B > A > D > C$
4	$B > C > A > D$

Perfectly Balanced, As All Things Should Be

Thm: Algorithm output pessimal for companies

Proof:

- Let output pair S with C
- Suppose $\exists$ stable pairing with (S', C), S' $\leq$ S
- Let C' be S' company in that pairing
- C optimal for S, so C' $\leq$ C
- Then (C, S) is rogue — contradiction!

Fin

Good luck on the final!