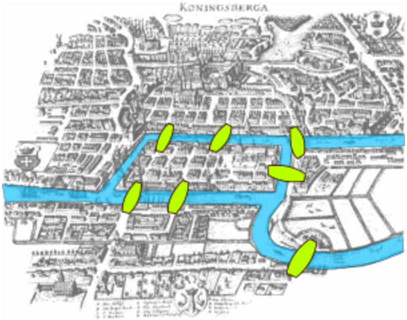
Lecture 4: Graph Theory 1

In Which We Draw a Bunch Of Pretty Pictures

The Seven Bridges of Königsberg



Source: Wikipedia

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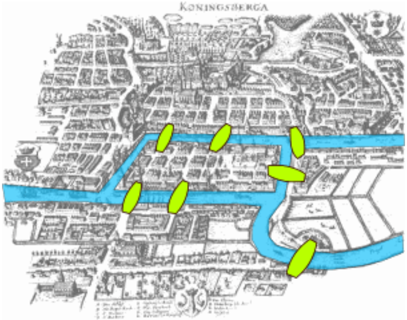

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Cross each bridge once and end where you started?

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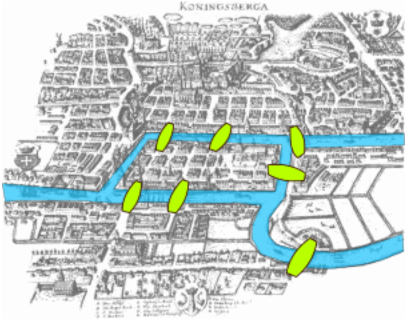

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Cross each bridge once and end where you started?

Turns out, impossible! Proved by Euler in 1736, inventing graph theory to do so.

What Is a Graph?

A *graph* $G = (V, E)$ is:

- A set of *vertices*[1] V
- A set of *edges* $E \subseteq V \times V$

[1] Sometimes also known as *nodes*

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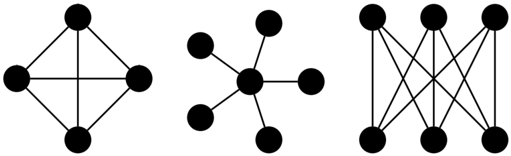

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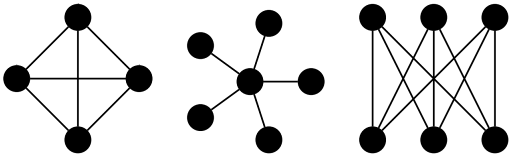

Not the graph of a function!

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Can You Give Me Directions?

Edges model relationships between vertices

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Focus (mainly) on undirected graphs in 70

Terminology

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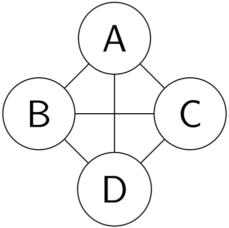
For a vertex v:

- number of edges incident is the *degree*
- u is a *neighbor* (or is *adjacent*) if $\{u, v\} \in E$
- if no neighbors, v is *isolated*

Paths and Cycles and Tours, Oh My!



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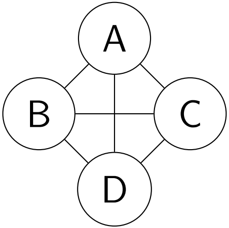

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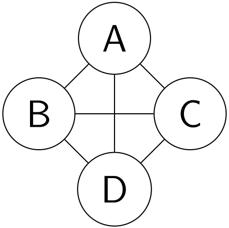

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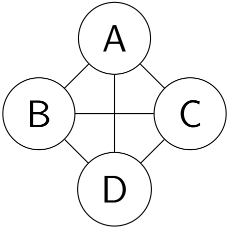

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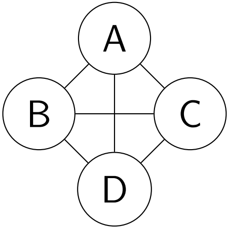

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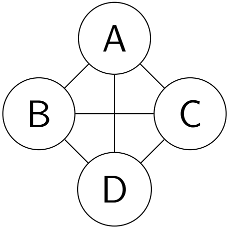

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Cycle $\subseteq$ tour $\subseteq$ walk; path $\subseteq$ walk.

Eulerian Tours

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- But all incident edges used!
- Thus, degree of v is even

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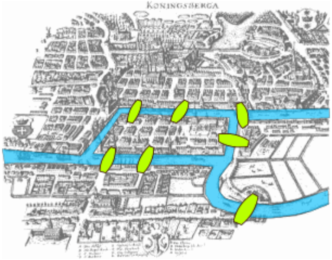
- Remove this tour, recurse on *connected components*
- "Splice" the recursive tours into the main one
- Result is Eulerian Tour of G!

Back To Königsberg



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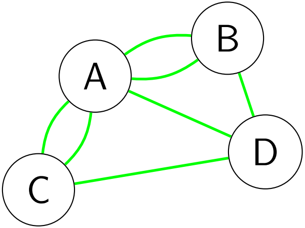

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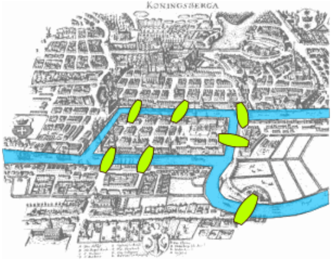

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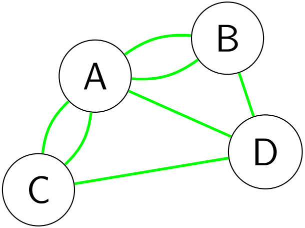

Model Königsberg as a graph
"Cross each bridge once" $\equiv$ "Eulerian Tour"

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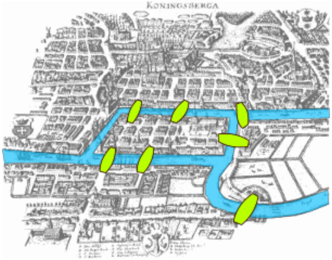

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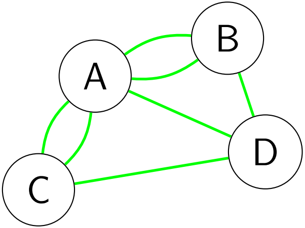

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"Cross each bridge once" $\equiv$ "Eulerian Tour"

Note: This is a *multigraph*
Everywhere else, assume *simple graphs*

- No repeated edges
- No self-loops

if(tired) { break; }

Whew, that was a long proof. Time for a break.

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Today's Discussion Question:
What building on campus would you get rid of?

Special Types of Graphs

Complete graphs have every possible edge

- Denote complete graph on n vertices as K_n
- K_5 has an important role next lecture!

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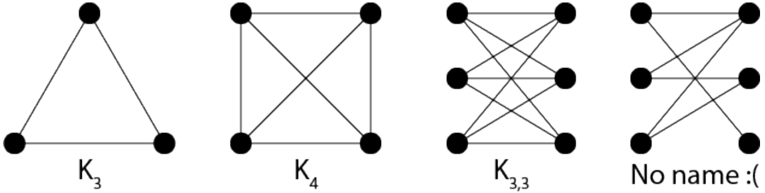
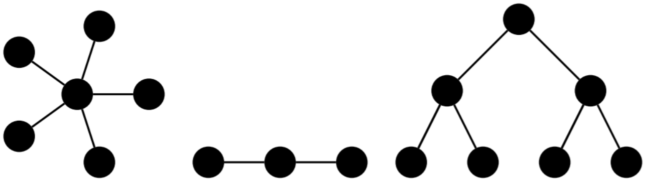


Can't See the Forest For All the Trees

A *tree* is a connected, acyclic graph



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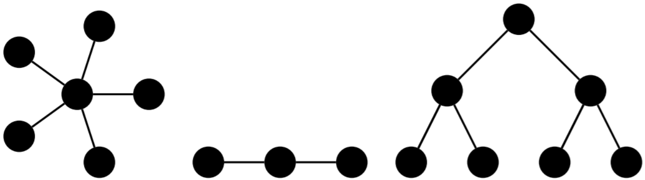

Equivalent definitions:

- Connected and $|V| - 1$ edges
- Connected and any edge removal disconnects
- Acyclic and any edge addition creates cycle

Leaf Lemmas

A *leaf* is a vertex of degree 1

Lemma: Every tree has at least one leaf.

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Allows us to do induction on trees!

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- So original graph had $|V| - 1$ edges

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- By IH, result is acyclic
- Adding v back can't create a cycle or disconnect!

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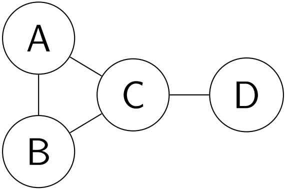

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See what happens if we use this in our "proof"

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We're stuck!
And should be–the theorem is false

Fin

Next time: moar graphs!