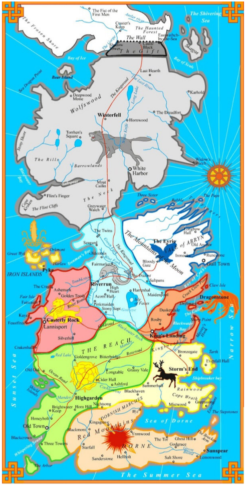
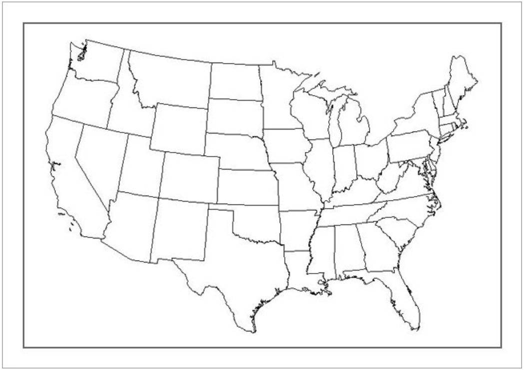
Lecture 5: Graph Theory 2

Snakes On a Planar Graph

Coloring a Map

How many colors required for this map?

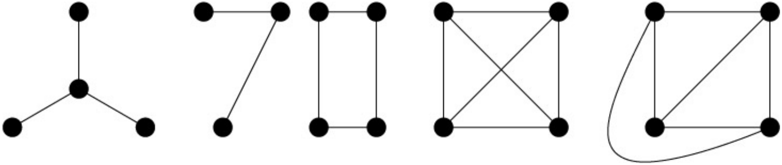




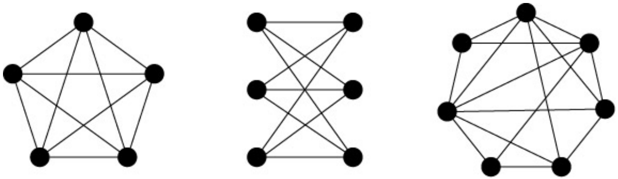
Planar Graphs

Graph is *planar* if can be drawn w/o edge crossings

Examples:

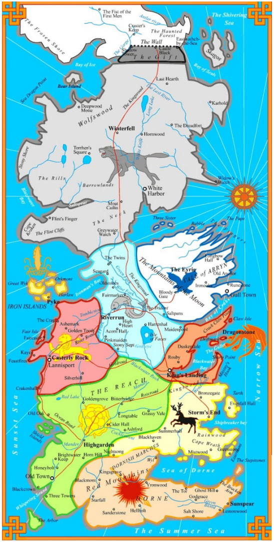


Not Examples:



But Whhhhyyyyyy???

Why do we care about planar graphs?



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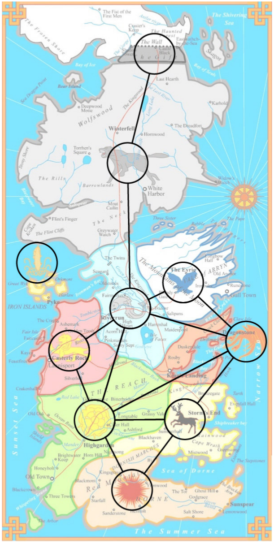
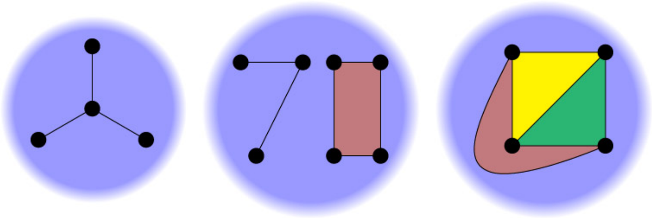


Face(book)

A *face* is connected region of plane

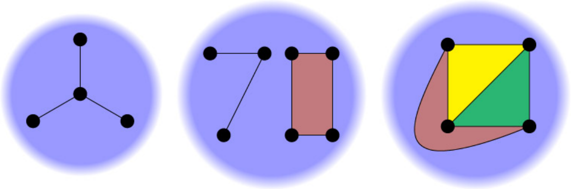


Claim: Conn. graph has one face $\iff$ is a tree

Intuition: have interior face $\iff$ have cycle

The Return Of the Euler

Theorem: For a conn. planar graph, $v + f = e + 2$.[1]
Let's verify this on example graphs



1st one: $v = 4$, $e = 3$, $f = 1$ ✓
2nd one, first half: $v = 3$, $e = 2$, $f = 1$ ✓
2nd one, second half: $v = 4$, $e = 4$, $f = 2$ ✓
3rd one: $v = 4$, $e = 6$, $f = 4$ ✓

[1] This is known as Euler's formula

Proof Of Euler

Theorem: For a conn. planar graph, $v + f = e + 2$.

Proof:

- By induction on f
- Base Case ($f = 1$): tree, so $e = v - 1$
  Thus $e + 2 = v + 1 = v + f$
- Suppose true for k faces
- For $k + 1$, remove edge between two faces
- k faces, so $v + k = (e - 1) + 2$
- Add 1 to both sides: $v + f = e + 2$

Sparsity

Euler: planar graphs have few edges.

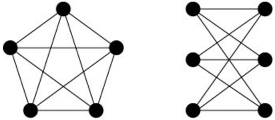
Theorem: For conn. planar graph, $e \leq 3v - 6$.

Proof:

- Each edge has 2 "sides" ($s = 2e$)
- Each face has ≥ 3 "sides" ($s \geq 3f$)
- Thus, $2e \geq 3f$, so $f \leq \frac{2}{3}e$
- Euler: $v + f = e + 2$
- Plug in for f: $v + \frac{2}{3}e \geq e + 2$
- Thus $\frac{1}{3}e + 2 \leq v$, so $e \leq 3v - 6$

Non-Planarity

Claim: K_5 and $K_{3,3}$ are non-planar.



For K_5, $e = 10$, but $3v - 6 = 3(5) - 6 = 9$!

$K_{3,3}$ has $e = 9$ and $3v - 6 = 3(6) - 6 = 12$
Not enough information to prove for $K_{3,3}$ yet!

Bipartite Planarity

Theorem: Bipartite planar graph has $e \leq 2v - 4$.

Proof:

- As before, edges have two sides ($s = 2e$)
- Bipartite means no triangles! So $s \geq 4f$
- Hence $2e \geq 4f$, so $f \leq \frac{1}{2}e$
- Plug into Euler's: $v + \frac{1}{2}e \geq e + 2$
- Thus $\frac{1}{2}e + 2 \leq v$, so $e \leq 2v - 4$

For $K_{3,3}$, $2v - 4 = 2(6) - 4 = 8$
9 edges means non-planar!

Why K_5 and $K_{3,3}$?

Kuratowski's Theorem: A graph is non-planar iff it "contains" K_5 or $K_{3,3}$.

Full meaning of "contains" beyond our scope
Less general: non-planar if has exact copy

What Were We Talking About Again?

Back to coloring!

Theorem: Any planar graph can be 6-colored.

To prove, need following lemma:
Every planar graph has a degree ≤ 5 vertex.

Proof:

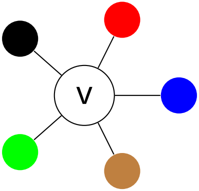
- Previously: $e \leq 3v - 6$
- Total degree is $2e \leq 6v - 12$
- Thus average degree is $\leq \frac{6v-12}{v} < 6$
- Not every vertex above average!

6-Color Theorem

Theorem: Any planar graph can be 6-colored.

Proof:

- By induction on $|V|$
- Base Case ($|V| = 1$): only need 1 color...
- Suppose true for graphs on k vertices
- Take G on $k + 1$ vertices
- Remove v st $\deg(v) \leq 5$, 6-color result
- v has ≤ 5 neighbors, so color available!



Zzzzzzzz...

Break time–be social!

Today's Discussion Question:
What vegetable or fruit would you be and why?

5-Color Theorem

Theorem: Any planar graph can be 5-colored.

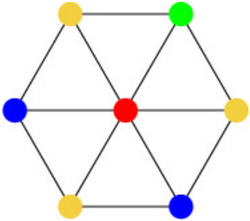
Proof:

- Same idea as 6-color theorem
- Remove deg ≤ 5 vertex, color, add back
- If deg ≤ 4, color remaining, so fine
- If two neighbors same color, again fine
- Problem if all 5 neighbors have different color
- Need to modify original coloring to fix!

Missed Connections

Will consider *color connected components*[2]

Idea: remove all verts not colored c_1 or c_2 from G
For vertex v colored c_1 or c_2, CCC(G, v, c_1, c_2) is connected component in result that contains v

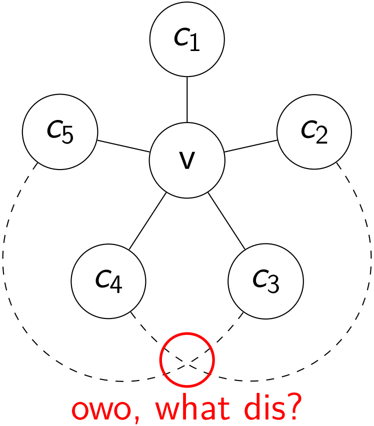


Claim: can reverse colors in any CCC and be fine

[2] This is totally not a term I just made up *looks around shiftily*

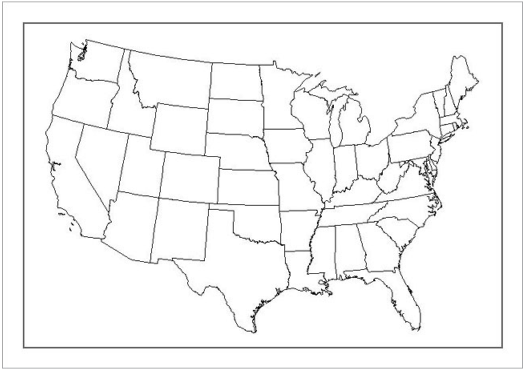
Back To 5-Coloring

Fix a planar drawing and recursive coloring:



Try to change c_5 to c_3
Try to change c_4 to c_2

Bringing It Back



This map can be colored with 5 colors!

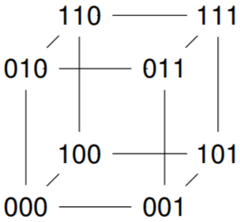
In fact, is a 4-color theorem as well.
Computer aided proof, not yet human readable.

Hypercubes

One more special type of graph: hypercubes!

Intuition: few edges, but "hard" to cut in half
Good design for communication network!

Formal definition: n-dimensional hypercube has vertex for each length-n bitstring
Edge between vertices iff they differ in one bit

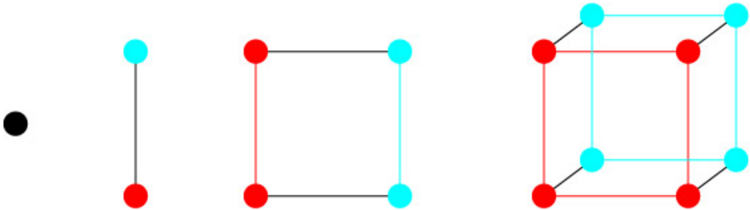


A Recursive Definition

Alernately define hypercubes by recursion:

0-dimensional hypercube is single vertex

$(n+1)$-dim hypercube is two copies of n-dim
Corresponding vertices connected by edges



What Does That Even Mean?

Claim: hypercube is “hard” to cut in half.
What does this mean, formally?

Theorem: To separate hypercube into sets S_1 and S_2, need to cut $\geq \min(|S_1|, |S_2|)$ edges.

Intuition: maybe easy to cut off a few vertices, hard to cut off a lot.

Proof in notes if you're interested ;)

Fin

Next time: modular arithmetic!