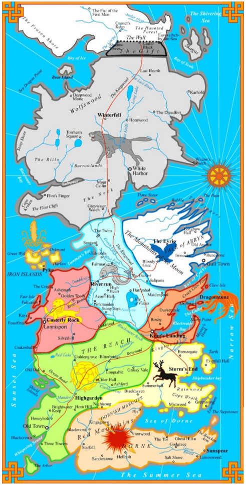
Lecture 5: Graph Theory 2

Snakes On a Planar Graph

Coloring a Map

How many colors required for this map?



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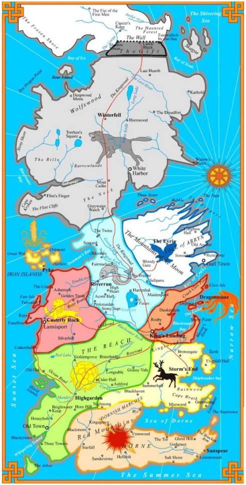

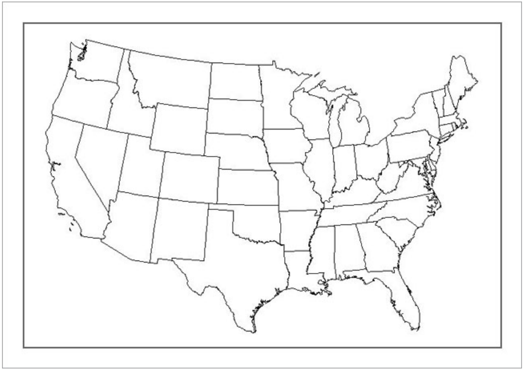

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Graph is *planar* if can be drawn w/o edge crossings

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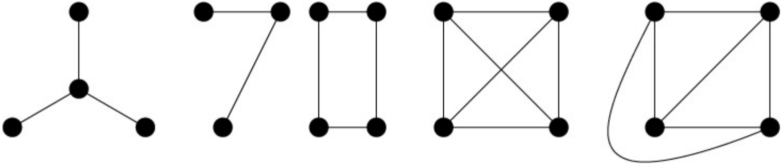

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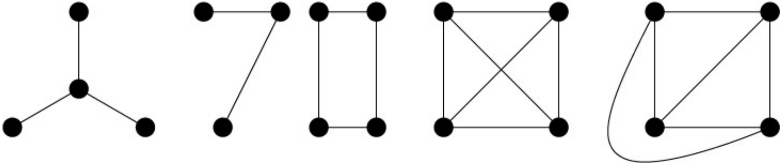

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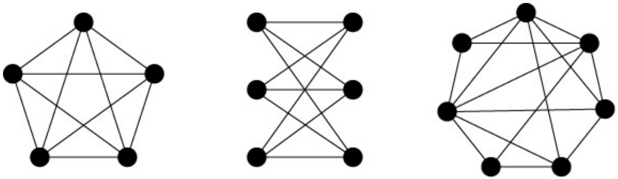

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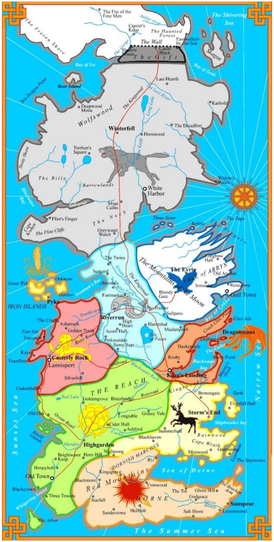

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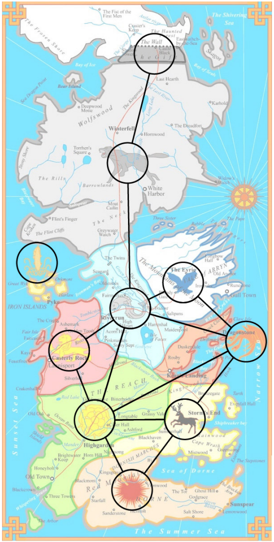

Face(book)

A *face* is connected region of plane

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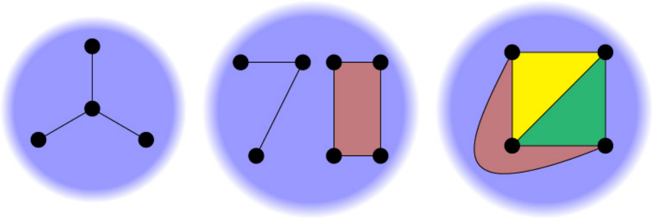

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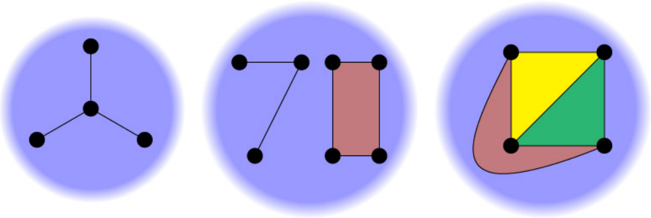

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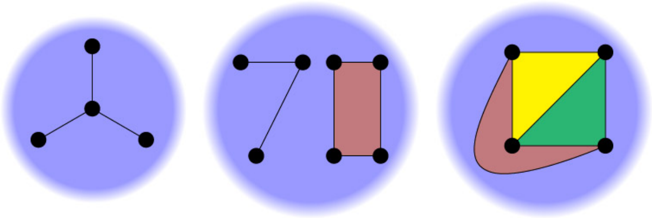

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Intuition: have interior face $\iff$ have cycle

The Return Of the Euler

Theorem: For a conn. planar graph, $v + f = e + 2$.[1]

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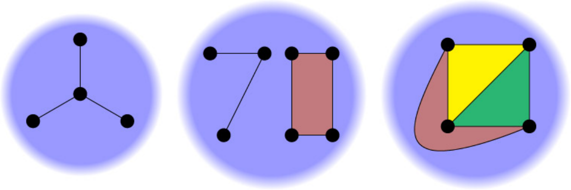

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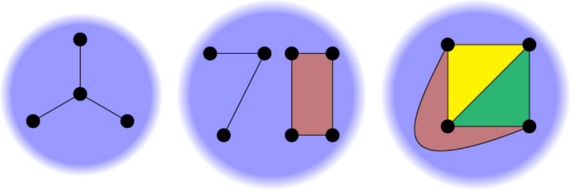

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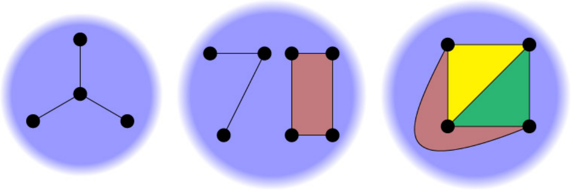

1st one: $v = 4$, $e = 3$, $f = 1$ ✓
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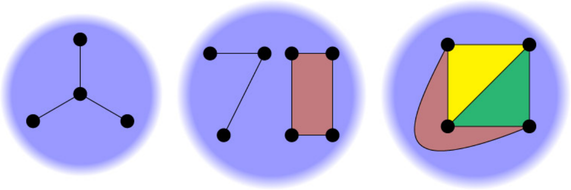

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3rd one: $v = 4$, $e = 6$, $f = 4$ ✓

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- Add 1 to both sides: $v + f = e + 2$

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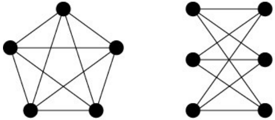
Non-Planarity

Claim: K_5 and $K_{3,3}$ are non-planar.



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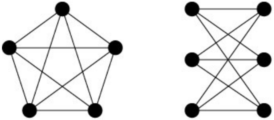

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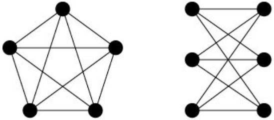

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$K_{3,3}$ has $e = 9$ and $3v - 6 = 3(6) - 6 = 12$
Not enough information to prove for $K_{3,3}$ yet!

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For $K_{3,3}$, $2v - 4 = 2(6) - 4 = 8$
9 edges means non-planar!

Why K_5 and $K_{3,3}$?

Kuratowski's Theorem: A graph is non-planar iff it "contains" K_5 or $K_{3,3}$.

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Kuratowski's Theorem: A graph is non-planar iff it "contains" K_5 or $K_{3,3}$.

Full meaning of "contains" beyond our scope
Less general: non-planar if has exact copy

What Were We Talking About Again?

Back to coloring!

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- Not every vertex above average!

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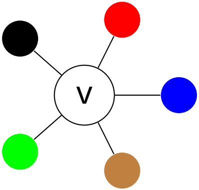
- v has ≤ 5 neighbors, so color available!



Zzzzzzzz...

Break time–be social!

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Today's Discussion Question:
What vegetable or fruit would you be and why?

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- Problem if all 5 neighbors have different color
- Need to modify original coloring to fix!

Missed Connections

Will consider *color connected components*[2]

[2] This is totally not a term I just made up *looks around shiftily*

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For vertex v colored c_1 or c_2, $\text{CCC}(G, v, c_1, c_2)$ is connected component in result that contains v

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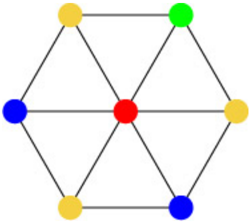

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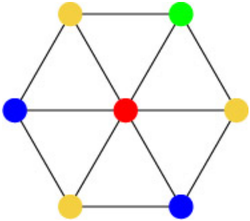

Claim: can reverse colors in any CCC and be fine

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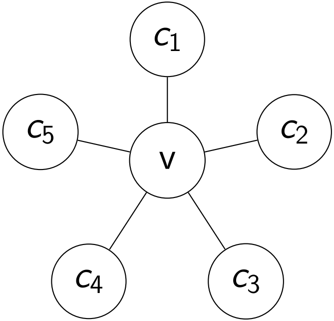
Back To 5-Coloring

Fix a planar drawing and recursive coloring:



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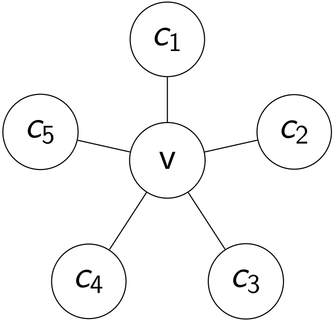

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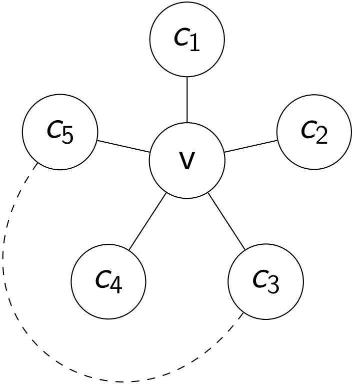

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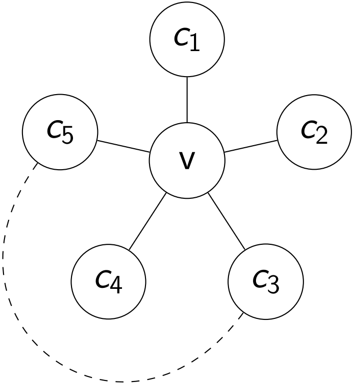

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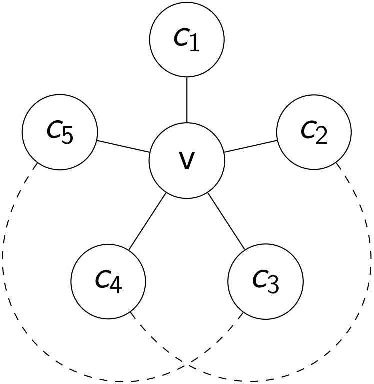

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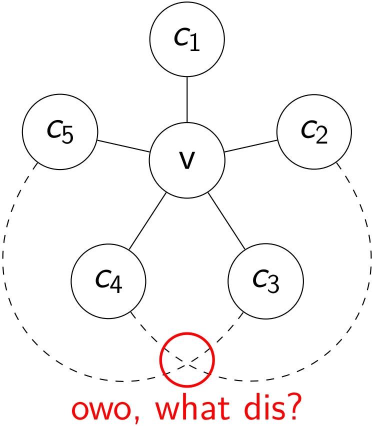

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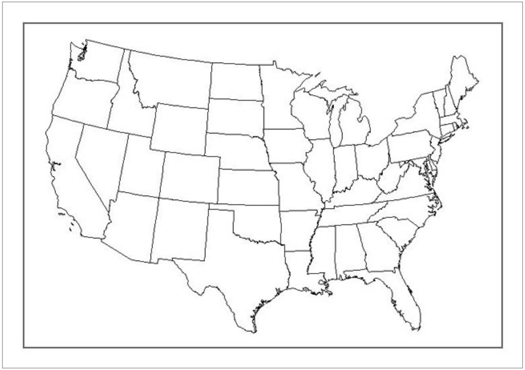

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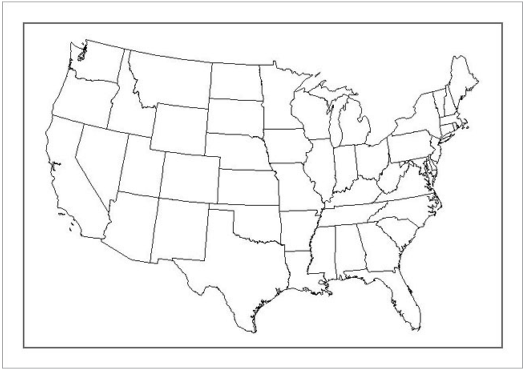

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In fact, is a 4-color theorem as well.
Computer aided proof, not yet human readable.

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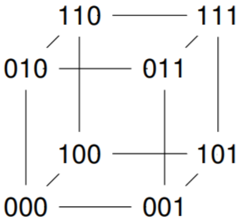

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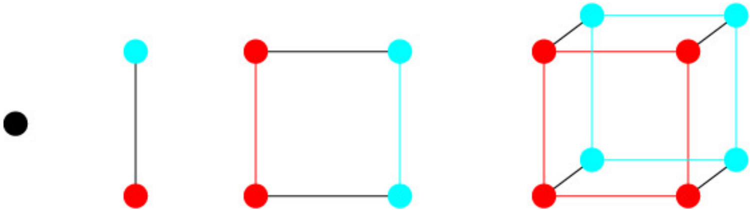

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Intuition: maybe easy to cut off a few vertices, hard to cut off a lot.

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What does this mean, formally?

Theorem: To separate hypercube into sets S_1 and S_2, need to cut $\geq \min(|S_1|, |S_2|)$ edges.

Intuition: maybe easy to cut off a few vertices, hard to cut off a lot.

Proof in notes if you're interested ;)

Fin

Next time: modular arithmetic!