Lecture 8: Cryptography

Trust No One.

Cryptography: Basic Set Up

Goal: system st Bob gets the message, Eve doesn't

XOR

First scheme built on the XOR operation:

x	y	x ⊕ y
0	0	0
0	1	1
1	0	1
1	1	0

Claim: $(x \oplus b) \oplus b = x$ for any bits x, b
$b = 0$ doesn't flip, $b = 1$ flips twice

One-Time Pad

Alice wants to send an n-bit message m to Bob

Setup:

- A and B generate random n-bit pad p

Encryption:

- A creates ciphertext $c = E_p(m) := m \oplus p$

Decryption:

- B decrypts $m = D_p(c) := c \oplus p$

Does Bob receive the message correctly?
Can Eve read the message?

OTP Correctness

Claim: Bob always receives the message Alice sent.

Formally: $\forall$ messages m & pads p, $D_p(E_p(m)) = m$

Proof:

- $E_p(m) = m \oplus p$, so $D_p(E_p(m)) = (m \oplus p) \oplus p$
- Each bit of m XORed by same bit twice
- By previous claim, each bit of m stays the same
- Thus $D_p(E_p(m)) = m$

OTP Security

Claim: Any message possible just given ciphertext.

Formally: $\forall$ c & m, $\exists$ pad p st $E_p(m) = c$
Proof:

- Take $p = c \oplus m$
- Then $E_p(m) = p \oplus m = (c \oplus m) \oplus m = c$

Intuition: set $p_i = 1$ iff ith bit needs to flip

w/o pad, c says nothing about m!

Problems With OTP

How do Alice and Bob agree on their pad?
Can't just send it over the channel!

Secure only for a single message — can't reuse pad!

Solve these issues with *public key cryptography*

Idea: don't assume shared secret key
Have separate private (only Bob) and public keys

"Textbook" RSA Protocol

Alice wants to send an n-bit message m to Bob

Setup:

- B chooses primes p, q st $pq > 2^n$
- B chooses e st $\gcd(e, (p-1)(q-1)) = 1$
- B publicizes $N = pq$ and e
- B keeps p, q, $d = e^{-1} \pmod{(p-1)(q-1)}$

Encryption:

- A encrypts $c = E_{N,e}(m) := m^e \pmod{N}$

Decryption:

- B decrypts $m = D_{N,d}(c) := c^d \pmod{N}$

Fermat's Little Theorem

Theorem: Let p be a prime and $a \not\equiv 0 \pmod{p}$.
Then $a^{p-1} \equiv 1 \pmod{p}$.

Proof:

- Consider set $S_p = \{1, 2, 3, \dots, p-1\}$
- Claim: $f(x) = ax \pmod{p}$ is bijection $S_p \to S_p$
- $\{1, 2, \dots, p-1\} = \{a, 2a, \dots, (p-1)a\} \pmod{p}$
- Means $\prod_i i \equiv \prod_i ia \equiv a^{p-1} \prod_i i \pmod{p}$
- Multiply by $\prod_i i^{-1}$, get $1 \equiv a^{p-1} \pmod{p}$

Proof Of Claim

To finish FLT proof, need to prove:
Claim: $f(x) = ax \pmod{p}$ is bijection $S_p \to S_p$
Proof:

- Need that for $x \in S_p$, $f(x) \in S_p$
  - If $x \in S_p$, $p \not| x$
  - $p \not| a$ either, so $p \not| ax$
  - Hence $ax \pmod{p} \in S_p$
- Inverse is $f^{-1}(y) = a^{-1}y \pmod{p}$
  - $f^{-1}(f(x)) \equiv a^{-1}ax \equiv x \pmod{p}$
  - $f(f^{-1}(x)) \equiv aa^{-1}x \equiv x \pmod{p}$

RSA Correctness

Theorem: RSA protocol always decrypts correctly.

Formally: $\forall$ p, q, e, and m, $D_{N,d}(E_{N,e}(m)) = m$

Proof:

- Note: $D(E(m)) = m^{ed} \bmod N$
- So just need to prove $m^{ed} \equiv m \pmod{N}$
- $ed = 1 + k(p-1)(q-1)$
- So $m^{ed} = (m^{(p-1)})^{k(q-1)} m \equiv m \pmod{p}$
- Similarly, have $m^{ed} \equiv m \pmod{q}$
- $m^{ed} \equiv m \pmod{pq}$ is solution to those two
- CRT: m is *only* solution!

RSA Efficiency

Need protocol to run quickly
For security, p and q often 512 bits or more.

Setup: need to sample p and q (next slide)
Setup: need to invert e to get d

- EGCD runs in log time!

Encryption: need to find $m^e \pmod{N}$

- Repeated squaring runs in log time!

Decryption: need to find $c^d \pmod{N}$

- Again use repeated squaring!

Sampling Primes

How to find primes p and q?
Can't use the same ones for every key!

Theorem: Num primes $\leq n$ at least $\frac{n}{\ln(n)}$

Means we can guess randomly until we find one!
Note: can quickly test primality

Time For A Break

4 minute breather!

Today's Discussion Question:
What is the best kind of sandwich?

RSA Security

Correctness and efficiency great; need security too

Open problem in Computer Science!
Generally accepted as secure, but no proof (yet)

Can easily break if factor N into p and q
But naïve factoring too slow if p and q big

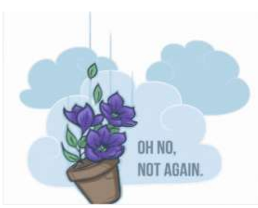
Note: can factor quickly on quantum computers
Not an immediate issue, but may be in the future!

Breaking Textbook RSA

Even if RSA secure, need careful implementation

Ex: suppose my credit card number is m
I send Amazon $E(m)$ to make a purchase

Alice can't recover m from $E(m)$...
...but what if she sends $E(m)$ to Amazon?



Defense Against Replay Attacks

Last slide was a *replay attack*

Fix: pad message with a bunch of randomness
If Amazon gets same message twice, reject

Moral: even secure protocol can be vulnerable!

Digital Signature Scheme

Alternate use of RSA: proof of identity

"Amazon" wants to send me a message.
How do I know it's actually Amazon?

Idea: Amazon sends $s = m^d \pmod{N}$ along with m
I can verify $s^e \equiv m \pmod{N}$

Only Amazon can sign consistently!
Ability to sign $\equiv$ ability to decrypt

Digital Signature Attack

Eve: I choose message to sign to prevent cheating!
Amazon: ok...
Eve: Sign $r^e E(m)$ pls
Amazon: $(r^e E(m))^d \pmod{N}$

What can Eve now do?
$(r^e E(m))^d \equiv r^{ed} m^{ed} \equiv rm \pmod{N}$

Uh oh — Eve knows r, so can invert to get m!

Moral: don't sign arbitrary messages

Fin

Next time: polynomials!